Classification of (low rank) Modular Tensor Categories, Part III

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today

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- The (semi)ring generated by the X_i with $\otimes, \oplus$ is called the *fusion algebra* of the category.

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- X_0 is special. $N^0 = I$, and $\theta_0 = 1$.
- There is an S matrix as well, but defining it will take too long. We'll list some of its properties later.
- The pair (S, T) is called the modular data of the category.

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A bit on modularity

The reason the adjective modular appears here is that S and T define a r-dimensional projective representation of $\mathrm{SL}(2,\mathbb{Z})$ that factors through $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$. This is called a level N representation.

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Finally, we have that the matrix obtained from S by dividing column i by d_i simultaneously diagonalizes the N^k. We will call this matrix $\tilde{S}$.

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Why?

Any modular category has admissible modular data. It is thought that all admissible modular data actually occurs as the modular data of some modular category. Thus, as a first step, it's a good idea to find all the data (it doesn't determine the category uniquely)

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- Representation Theory of $\mathrm{SL}(2, \mathbb{Z})$
  - Constraints on both the order and spectra of the T-matrix
- Computational Algebra
  - Gröbner Basis Algorithm (Maple and Macaulay 2)
  - Wolfram Mathematica

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$$S_{ij} = \epsilon_\sigma(i)\epsilon_(j)S_{\sigma(i)\sigma^{-1}(j)}$$

$$\prod_{j=0}^{r-1} \epsilon_\sigma(j) = (-1)^\sigma \qquad \text{(for even } r\text{)}$$

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$$\sigma(\theta_i) = \theta_{\sigma^2(i)}$$

Current case

We're interested in $r = 6$, $\mathrm{Gal}(S) = \langle (012)(345) \rangle$.

The answers:

Theorem

Up to relabeling, and Galois conjugation, the only modular data for rank 6, self dual, MTC's with Galois group $\langle(012)(345)\rangle$ are given by the following 2 pairs of (S, T).

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & d & d^2-1 \\ d & -(d^2-1) & 1 \\ d^2-1 & 1 & -d \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & \\ & i \end{bmatrix} \otimes \begin{bmatrix} 1 & & \\ & e^{2\pi i/7} & \\ & & e^{10\pi i/7} \end{bmatrix}$$

where $d = 2\cos(\pi/7)$ *and*

The answers:

$$S = \begin{bmatrix} 1 & -1 & 1 & r_1 & r_2 & r_3 \\ -1 & 1 & -1 & -r_2 & -r_3 & -r_1 \\ 1 & -1 & 1 & r_3 & r_1 & r_2 \\ r_1 & -r_2 & r_3 & 1 & 1 & 1 \\ r_2 & -r_3 & r_1 & 1 & 1 & 1 \\ r_3 & -r_1 & r_2 & 1 & 1 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & & & & & \\ & e^{4\pi i/3} & & & & \\ & & e^{2\pi i/3} & & & \\ & & & e^{2\pi ik/9} & & \\ & & & & e^{2\pi i/9} & \\ & & & & & e^{8\pi i/9} \end{bmatrix}$$

where with α a primitive 18th root of unity,
$r_1 = -\alpha - \alpha^2 + \alpha^5$, $r_2 = \alpha + \alpha^2 - \alpha^4$ and $r_3 = \alpha^4 - \alpha^5$.

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A symmetry that actually produces a solution:

$$\begin{bmatrix} 1 & d_1 & d_2 & d_3 & d_4 & d_5 \\ d_1 & d_2 & -1 & -d_4 & -d_5 & -d_3 \\ d_2 & -1 & -d_1 & d_5 & d_3 & d_4 \\ d_3 & -d_4 & d_5 & S_{33} & S_{34} & S_{35} \\ d_4 & -d_5 & d_3 & S_{34} & S_{35} & S_{33} \\ d_5 & -d_3 & d_4 & S_{35} & S_{33} & S_{34} \end{bmatrix}$$

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Factored Polynomials	Zero Factors Added
$p(d_3 - d_4 + d_5)D$	$d_3 - d_4 + d_5$
$D^2(p^2 - D^2), D^4(\theta_3 + \theta_4 + \theta_5 + 1)$	$p^2 - D^2, \theta_3 + \theta_4 + \theta_5 + 1$
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Note: This won't work in the (012) case. We know of a solution where the last three entries of the T matrix are the same.

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The fact that either $7|N$ or $9|N$ comes from Proposition 3.13 in *BNRW* and will continue to hold in the (012) case. It's essentially a consequence of the fact that $\mathbb{Q}(T)/\mathbb{Q}(S)$ is a 2-group.

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$9|N$ means we have fusion rules like B_9.

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So we have a product category. [RSW] classifies all the 2 and 3 dimensional categories, so we can just look for the right modular data there.

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We solve for the top left corner of S, which gives all of S. Once we have that, Gröbner bases give enough relations to solve for all of T, and we're done.

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- Sort out the root of unity issue in T.
- See if this implies we know the categories.
- Sleep during normal hours.

References

BNRW : P. Bruillard, S. Ng, E. Rowell, Z. Wang; On Classification of Modular Categories By Rank

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