Dedekind Sums Arising from Generalized Eisenstein Series

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Abstract

Given primitive Dirichlet characters χ_1 and χ_2, we study the weight zero Eisenstein series $E_{\chi_1,\chi_2}(z,s)$ at $s=1$. We examine transformation properties of terms arising from the Fourier expansion of the Eisenstein series, and we express these properties with a generalized Dedekind sum formula in terms of Bernoulli functions.

1 Introduction

Let χ_1, χ_2 be primitive Dirichlet characters modulo q_1, q_2, respectively, with $\chi_1(-1)\chi_2(-1) = -1$. We investigate the generalized Dedekind sum arising from the weight-zero Eisenstein series attached to characters. The Eisenstein series is defined as

$$E_{\chi_1,\chi_2}(z,s) = \frac{1}{2} \sum_{(c,d)=1} \frac{(q_2 y)^s \chi_1(c)\chi_2(d)}{|cq_2 z + d|^{2s}}.$$

Here E_{χ_1,χ_2} is an automorphic form on the congruence subgroup $\Gamma_0(q_1q_2)$ of nebentypus $\psi = \chi_1\overline{\chi_2}$. Precisely, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, $E_{\chi_1,\chi_2}(\gamma z, s) = \psi(\gamma)E_{\chi_1,\chi_2}(z,s)$, where $\psi(\gamma) = \psi(d)$.

We are investigating the Eisenstein series at $s=1$. Historically, many authors have been interested in the Eisenstein series as $s \to 1$ because it is has been a way to arrive at the Dedekind sum. Additionally, the first Kronecker Limit Formula involves looking at $\lim_{s\to 1}$ of the Eisenstein series with trivial Dirichlet characters, and investigating the Eisenstein series at $s=1$ leads to a generalization of the first Kronecker Limit Formula. For example, Goldstein [3] finds a generalized Eisenstein series of the first Kronecker Limit Formula by looking at the Eisenstein series defined at a cusp.

Specifically, we want to look at the Fourier expansion of E_{χ_1,χ_2} at $s=1$. It is more convenient to consider the "completed" Eisenstein series defined by

$$E^*_{\chi_1,\chi_2}(z,s) := \frac{(q_2/\pi)^s}{i^{-k}\tau(\chi_2)} \Gamma(s + \frac{k}{2}) L(2s, \chi_1\chi_2) E_{\chi_1,\chi_2}(z,s).$$

Here τ denotes the Gauss sum given by

$$\tau(\chi) = \sum_{n=0}^{q-1} \chi(n) e^{\frac{2\pi i n}{q}},$$

for χ modulo q.

The Fourier expansion for $E^*_{\chi_1,\chi_2}$ is conveniently stated by Young [7] (see also Huxley [5]). When $q_1, q_2 \neq 1$, the Fourier expansion simplifies as

$$E^*_{\chi_1,\chi_2}(z,s) = 2\sqrt{y} \sum_{n\neq 0} \lambda_{\chi_1,\chi_2}(n,s) e(nx) K_{s-\frac{1}{2}}(2\pi|n|y),$$

where

$$\lambda_{\chi_1,\chi_2}(n,s) = \chi_2(\mathrm{sgn}(n)) \sum_{ab=|n|} \chi_1(a)\overline{\chi_2}(b) \left(\frac{b}{a}\right)^{s-\frac{1}{2}},$$

and $e(x) = e^{2\pi i x}$.

When $s = 1$, the Fourier expansion simplifies as

$$E^*_{\chi_1,\chi_2}(z,1) = f_{\chi_1,\chi_2}(z) + \chi_2(-1)\overline{f_{\overline{\chi_1},\overline{\chi_2}}(z)}, \tag{1.1}$$

where

$$f_{\chi_1,\chi_2}(z) = \sum_{n>0} \frac{e(nz)}{\sqrt{n}} \sum_{ab=n} \chi_1(a)\overline{\chi_2}(b) \left(\frac{b}{a}\right)^{\frac{1}{2}}.$$

Our generalized Dedekind sum arises from studying the transformation properties of f_{χ_1,χ_2} on $\Gamma_0(q_1q_2)$. Other authors, such as Nagasaka [6] and Goldstein and Razar [4], have studied functions similar to f_{χ_1,χ_2} to arrive at a generalized Dedekind sum. The process we use is unique because our function f_{χ_1,χ_2} arises naturally from the Fourier expansion of the Eisenstein series. To obtain our Dedekind sum, we study the function ϕ_{χ_1,χ_2}, defined for $\gamma \in \Gamma_0(q_1q_2)$ and $z \in \mathbb{H}$ as

$$\phi_{\chi_1,\chi_2}(\gamma, z) := f_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) f_{\chi_1,\chi_2}(z).$$

In Lemma 2.1, we show that ϕ_{χ_1,χ_2} is independent of z.

Let B_1 denote the first Bernoulli function given by

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}.$$

The following theorem gives our generalized Dedekind sum in terms of the Bernoulli function.

Theorem 1.1. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$. *Then*

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \,(\mathrm{mod}\ c)} \sum_{n \,(\mathrm{mod}\ q_1)} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} - \frac{aj}{c}\right),$$

where $c = c'q_1$.

Generalized Dedekind sums arising from the Eisenstein series have been studied by Berndt [1], Nagasaka [6], Goldstein [3], Dağh and Can [2], and others. Berndt and others began with a different version of the Eisenstein series. Our form of the Eisenstein series is more natural and has nice arithmetical properties. Our approach is unique because our generalized Dedekind sum follows from the Fourier expansion of the Eisenstein series, and we are able to calculate the generalized Dedekind sum directly from the transformation properties of the Eisenstein series.

2 Preliminary Results

Lemma 2.1. *The function* ϕ_{χ_1,χ_2} *is independent of* z*.*

Proof. Since $E^*_{\chi_1,\chi_2}(\gamma z,1) = \psi(\gamma)E^*_{\chi_1,\chi_2}(z,1)$ and $E^*_{\chi_1,\chi_2}(z,1) = f_{\chi_1,\chi_2}(z)+\chi_2(-1)\overline{f_{\overline{\chi_1},\overline{\chi_2}}}(z)$, it immediately follows that

$$\phi_{\chi_1,\chi_2}(\gamma,z) = -\chi_2(-1)\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}(\gamma,z). \tag{2.1}$$

Since ϕ_{χ_1,χ_2} is a holomorphic function and $\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}$ is an antiholomorphic function, ϕ_{χ_1,χ_2} must be constant. □

From now on, we will write $\phi_{\chi_1,\chi_2}(\gamma)$ instead of $\phi_{\chi_1,\chi_2}(\gamma,z)$.

For later reference, we state a more symmetric form for ϕ_{χ_1,χ_2}. Specifically, from (2.1) it follows that

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{1}{2}(\phi_{\chi_1,\chi_2}(\gamma) - \chi_2(-1)\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}(\gamma)). \tag{2.2}$$

In the following lemma, we investigate the homomorphic properties of ϕ_{χ_1,χ_2} that arise from the automorphic properties of $E^*_{\chi_1,\chi_2}$.

Lemma 2.2. *Let* $\gamma_1,\gamma_2 \in \Gamma_0(q_1q_2)$*. Then* $\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) = \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2)$*.*

Proof. Since ψ is multiplicative,

$$\begin{aligned}
\phi_{\chi_1,\chi_2}(\gamma_1\gamma_2) &= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1\gamma_2)f_{\chi_1,\chi_2}(z)\\
&= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1)\psi(\gamma_2)f_{\chi_1,\chi_2}(z)\\
&= f_{\chi_1,\chi_2}(\gamma_1\gamma_2 z) - \psi(\gamma_1)f_{\chi_1,\chi_2}(\gamma_2 z) + \psi(\gamma_1)f_{\chi_1,\chi_2}(\gamma_2 z) - \psi(\gamma_1)\psi(\gamma_2)f_{\chi_1,\chi_2}(z)\\
&= \phi_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2}(\gamma_2).
\end{aligned}$$

□

3 The Generalized Dedekind Sum

Our main goal is to find a finite sum formula for ϕ_{χ_1,χ_2}, which will give us our generalized Dedekind sum. Our process loosely follows the methodology of Goldstein [3]. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1q_2)$, and let $z = \frac{-d}{c} + \frac{i}{c^2u} \in \mathbb{H}$ for some $u \in \mathbb{R}$, $u \neq 0$. Then $\gamma z = \frac{a}{c} + iu$. Since ϕ_{χ_1,χ_2} is independent of z,

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u\to 0^+}\left(f_{\chi_1,\chi_2}\left(\frac{a}{c}+iu\right) - \psi(\gamma)f_{\chi_1,\chi_2}\left(\frac{-d}{c}+\frac{i}{c^2u}\right)\right).$$

From the Fourier expansion of $E^*_{\chi_1,\chi_2}$, it is clear that

$$\lim_{u\to 0^+} f_{\chi_1,\chi_2}\left(\frac{-d}{c}+\frac{i}{c^2u}\right) = 0.$$

Thus,

$$\phi_{\chi_1,\chi_2}(\gamma) = \lim_{u\to 0^+} f_{\chi_1,\chi_2}\left(\frac{a}{c}+iu\right). \tag{3.1}$$

To evaluate this limit, we begin simplifying f_{χ_1,χ_2} as

$$f_{\chi_1,\chi_2}(z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\chi_1(l)\overline{\chi_2}(k)}{l} e(klz). \tag{3.2}$$

Then

$$f_{\chi_1,\chi_2}(z) = \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \theta_{\chi_2}(z,l), \tag{3.3}$$

where

$$\theta_{\chi}(z,l) := \sum_{k=1}^{\infty} \overline{\chi}(k) e(klz).$$

for χ modulo q.

Lemma 3.1. *Let χ modulo q be a primitive character. Let $a, c, l \in \mathbb{Z}$ with $c \geq 1$, $c \equiv 0 \pmod{q}$, $(a,c) = 1$, and $l \not\equiv 0 \pmod{\frac{c}{q}}$. Then*

$$\lim_{u \to 0^+} \theta_{\chi}\left(\frac{a}{c} + iu, l\right) = \chi(-1) \sum_{j \pmod{c}} \overline{\chi}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right),$$

where B_1 is the first Bernoulli function given by

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Proof. One can write χ modulo q in terms of additive characters using the formula

$$\overline{\chi}(k) = \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) e\left(\frac{rk}{q}\right). \tag{3.4}$$

Applying this, we get

$$\theta_{\chi}(z,l) = \frac{1}{\tau(\chi)} \sum_{k=1}^{\infty} \sum_{r \pmod{q}} \chi(r) e\left(klz + \frac{rk}{q}\right). \tag{3.5}$$

When $z = \frac{a}{c} + iu$,

$$\theta_{\chi}\left(\frac{a}{c} + iu, l\right) = \frac{1}{\tau(\chi)} \sum_{k=1}^{\infty} \sum_{r \pmod{q}} \chi(r) e\left(\frac{akl}{c} + iukl + \frac{rk}{q}\right) \tag{3.6}$$

$$= \frac{1}{\tau(\chi)} \sum_{r \pmod{q}} \chi(r) \frac{e\left(\frac{al}{c} + iul + \frac{r}{q}\right)}{1 - e\left(\frac{al}{c} + iul + \frac{r}{q}\right)}. \tag{3.7}$$

As $u \to 0$, $e(iul) \to 1$. Thus,

$$\begin{aligned}\lim_{u\to 0^+} \theta_\chi\left(\frac{a}{c}+iu,l\right) &= \frac{1}{\tau(\chi)}\sum_{r(\mathrm{mod}\ q)} \chi(r)\frac{e\left(\frac{al}{c}+\frac{r}{q}\right)}{1-e\left(\frac{al}{c}+\frac{r}{q}\right)} \\ &= -\frac{1}{\tau(\chi)}\sum_{r(\mathrm{mod}\ q)} \chi(r)\frac{1}{1-e\left(-\frac{al}{c}-\frac{r}{q}\right)}.\end{aligned}$$

If $\eta \neq 1$ is a k^{th} root of unity, we have the following identity

$$\frac{1}{1-\eta} = -\frac{1}{k}\sum_{j=1}^{k-1} j\eta^j,$$

which is easily verified by multiplying each side of the equation by $1-\eta$. Since $q|c$, $e\left(-\frac{al}{c}-\frac{r}{q}\right)$ is a c^{th} root of unity. Applying this identity, we get

$$\lim_{u\to 0^+} \theta_\chi\left(\frac{a}{c}+iu,l\right) = \frac{1}{c\tau(\chi)}\sum_{r(\mathrm{mod}\ q)} \chi(r)\sum_{j=1}^{c-1} je\left(\frac{-alj}{c}-\frac{rj}{q}\right).$$

By (3.4), this gives us

$$\begin{aligned}\lim_{u\to 0^+} \theta_\chi\left(\frac{a}{c}+iu,l\right) &= \frac{\chi(-1)}{c\tau(\chi)}\sum_{j=1}^{c-1} je\left(\frac{-alj}{c}\right)\tau(\chi)\overline{\chi}(j) \\ &= \chi(-1)\sum_{j=1}^{c-1}\frac{j}{c}\overline{\chi}(j)e\left(\frac{-alj}{c}\right).\end{aligned}$$

Now we want to express this in terms of B_1. We have

$$\lim_{u\to 0^+} \theta_\chi\left(\frac{a}{c}+iu,l\right) = \chi(-1)\sum_{j(\mathrm{mod}\ c)} \overline{\chi}(j)\left(\frac{j}{c}-\left\lfloor\frac{j}{c}\right\rfloor-\frac{1}{2}+\frac{1}{2}\right)e\left(\frac{-alj}{c}\right).$$

We have

$$\overline{\chi}(j)\left(\frac{j}{c}-\left\lfloor\frac{j}{c}\right\rfloor-\frac{1}{2}\right) = B_1\left(\frac{j}{c}\right)$$

since $\overline{\chi}(j)=0$ when $\frac{j}{c}\in\mathbb{Z}$, so the expression simplifies as

$$\lim_{u\to 0^+} \theta_\chi\left(\frac{a}{c}+iu,l\right) = \left[\chi(-1)\sum_{j(\mathrm{mod}\ c)} \overline{\chi}(j)B_1\left(\frac{j}{c}\right)e\left(\frac{-alj}{c}\right)\right]+\left[\frac{\chi(-1)}{2}\sum_{j(\mathrm{mod}\ c)} \overline{\chi}(j)e\left(\frac{-alj}{c}\right)\right].$$

In the second bracketed term in the line above equals zero. To see this, write $j = A+qB$ where $A \pmod q$ and $B \pmod{c/q}$. Then

$$\sum_{j (\mathrm{mod}\ c)} \overline{\chi}(j) e\left(\frac{-alj}{c}\right) = \sum_{A (\mathrm{mod}\ q)} \overline{\chi}(A) e\left(\frac{-alA}{c}\right) \sum_{B (\mathrm{mod}\ c/q)} e\left(\frac{-alB}{c/q}\right).$$

Since $\frac{c}{q} \not| \, al$, the sum over B equals 0. Thus,

$$\lim_{u \to 0^+} \theta_\chi\left(\frac{a}{c} + iu, l\right) = \chi(-1) \sum_{j \pmod c} \overline{\chi}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right).$$

□

Now we simplify ϕ_{χ_1,χ_2} further using the function θ_χ.

Lemma 3.2. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$. *Then*

$$\phi_{\chi_1,\chi_2}(\gamma) = \chi_2(-1) \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j (\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right).$$

Proof. We apply Lemma 3.1 to 3.3.

$$\begin{aligned} \phi_{\chi_1,\chi_2}(\gamma) &= \lim_{u \to 0} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \theta_{\chi_2}\left(\frac{a}{c} + iu, l\right) \\ &= \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \lim_{u \to 0} \theta_{\chi_2}\left(\frac{a}{c} + iu, l\right) \\ &= \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \chi_2(-1) \sum_{j (\mathrm{mod}\ c)} \overline{B_1}\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right). \end{aligned}$$

□

Remark. The following theorem relies on the generalized Bernoulli function, which is stated by Berndt [1]. One may easily simplify Berndt's formulae to get the equivalent form

$$B_{1,\chi}(x) = \frac{-\tau(\overline{\chi})}{2\pi i} \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{\chi(l)}{l} e\left(\frac{lx}{q}\right) \tag{3.8}$$

for χ modulo q. We prefer using this form because it is more symmetric and does not depend on the parity of χ.

Proof of Theorem 1.1. By applying Lemma 3.2 to (2.2), we get the simplification below.

$$\begin{aligned} \phi_{\chi_1,\chi_2}(\gamma) &= \frac{\chi_2(-1)}{2} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j (\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right) \\ &\quad -\chi_2(-1) \frac{\chi_2(-1)}{2} \sum_{l=1}^{\infty} \frac{\chi_1(l)}{l} \sum_{j (\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{alj}{c}\right). \end{aligned}$$

Using the change of variables $l \to -l$ and the fact that $\chi_1(-1)\chi_2(-1) = 1$, this simplifies as

$$\begin{aligned}\phi_{\chi_1,\chi_2}(\gamma) &= \frac{\chi_2(-1)}{2} \sum_{l \neq 0} \frac{\chi_1(l)}{l} \sum_{j \,(\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e\left(\frac{-alj}{c}\right) \\ &= \frac{\chi_2(-1)}{2} \sum_{j \ (\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) \sum_{l \neq 0} \frac{\chi_1(l)}{l} e\left(\frac{-alj}{c}\right).\end{aligned}$$

Letting $c = c'q_1$ and substituting B_{1,χ_1} into this expression, we get

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \ (\mathrm{mod}\ c)} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_{1,\chi_1}\left(\frac{-aj}{c'}\right).$$

This sum may also be written as a finite expression using the following transformation formula by Berndt [1],

$$B_{1,\chi}(x) = \sum_{n=1}^{q-1} \overline{\chi}(n) B_1\left(\frac{x+n}{q}\right),$$

for χ modulo q.

Substituting this in, we get

$$\phi_{\chi_1,\chi_2}(\gamma) = \frac{-\pi i \chi_2(-1)}{\tau(\overline{\chi_1})} \sum_{j \,(\mathrm{mod}\ c)} \sum_{n \,(\mathrm{mod}\ q_1)} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} - \frac{aj}{c}\right).$$

□

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