On Roots of Polynomials over Prime Fields and the Roots of Unity

Tyler Feemster

Princeton University

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Definition and Directions

Beginning Goal

To determine when a non-trivial root exists over $\mathbb{F}_p$ of the polynomial

$$f(x) = \sum_{i=1}^{r} a_i x_i^{n_i},$$

where $a_i \in \mathbb{F}_p$, $x = (x_1, \ldots, x_r) \in \mathbb{F}_p^r$, and $n_i > 0$.

- The prime field $\mathbb{F}_p$ is the set of integers modulo p where addition, subtraction, multiplication, and division are well-defined via modular arithmetic.
- If $f(x) = 5 + 4x_1^2$ in $\mathbb{F}_7$, we have roots $x_1 = 2$ and $x_1 = 5$.

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Chevalley-Warning Theorem and Ax's Extension

Chevalley-Warning Theorem 1935

Let $f(x) = \sum_{i=1}^{r} a_i x_i^{n_i}$, where $a_i \in \mathbb{F}_p$, $x = (x_1, \ldots, x_r) \in \mathbb{F}_p^r$, and $n_i > 0$. If $deg(f) < r$, then $f(x)$ has 0 (mod p) roots.

Consider $x_1^2 + x_2^2 + x_3^2 = 0$ over $\mathbb{F}_{11}$. Since $(0, 0, 0)$ is a root, there must be at least 10 more.

Ax 1964

Let b be the largest positive integer strictly less than $r/deg(f)$. Then, $f(x)$ has 0 (mod p^b) roots.

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Condition for Guaranteed Root

Again, let $f(x) = \sum_{i=1}^{r} a_i x_i^{n_i}$ over $\mathbb{F}_p$.

- If there exists an n_i such that $gcd(n_i, p-1) = 1$, then there exists a non-trivial root automatically since $a_i x_i^{n_i}$ is a permutation of $\mathbb{F}_p$.
- If we consider the mapping $3x^5$ over $\mathbb{F}_7$, we obtain:

- Now, we see that $f(x) = 3x_1^5 + 4x_2^3$ has a root (7 actually).

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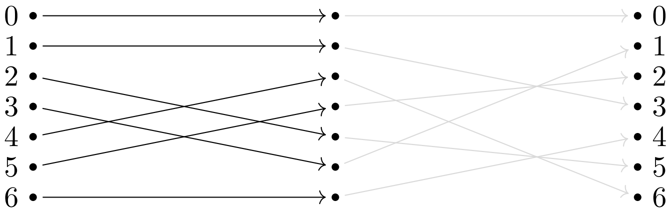

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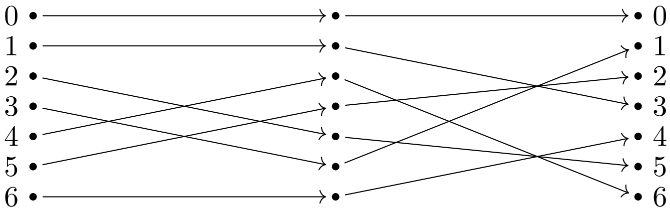

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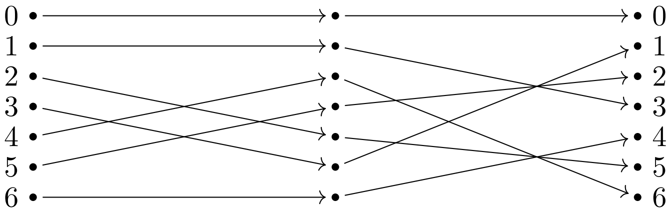

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Extending the Condition for Guaranteed Root

If $gcd(n_i, p-1) = gcd(n_j, p-1) = 2$ for some n_i and n_j, then the image of $a_i x_i^{n_i} + a_j x_j^{n_j}$ is $\mathbb{F}_p$.

- The image of $a_i x_i^{n_i}$ has exactly $\frac{p-1}{2} + 1$ elements in $\mathbb{F}_p$.
  - Follows from $n_i = 2m$ where x^m permutes $\mathbb{F}_p$.
- Given $b \in \mathbb{F}_p$, the image of $b - a_j x_j^{n_j}$ has $\frac{p-1}{2} + 1$ elements.
- The images of $b - a_j x_j^{n_j}$ and $a_i x_i^{n_i}$ have union of at most p elements, but $(\frac{p-1}{2} + 1) + (\frac{p-1}{2} + 1) = p + 1$.
- So, for some x_i and x_j, $b - a_j x_j^{n_j} = a_i x_i^{n_i}$.

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Pathological Polynomials

So, when are there no roots?

- Fermat's Little Theorem states that for any $x \in \mathbb{F}_p$, $x^{p-1} \in \{0, 1\}$, so $x^{\frac{p-1}{2}} \in \{-1, 0, 1\}$.
- Consider x^2 in $\mathbb{F}_5$:

$x_1^3 + x_2^3 - 3$ has no roots over $\mathbb{F}_7$,
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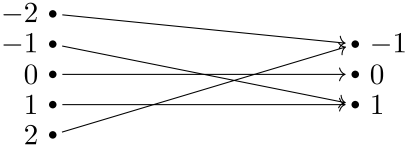

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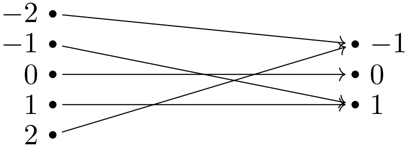

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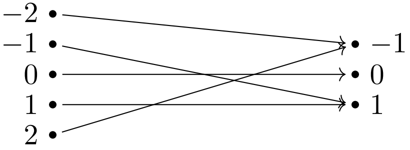

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Weil and his Bound

Weil 1949

Let $f(x) = \sum_{i=1}^{r} a_i x_i^{n_i}$, N be the number of roots of $f(x) + 1$, and $d_i = gcd(n_i, p-1)$. Then,

$$|N - p^{r-1}| \leq (d_1 - 1) \cdots (d_r - 1) p^{\frac{r-1}{2}}$$

- If $d_i = 1$ for any i, then $N = p^{r-1}$ exactly.
- If $d_i \geq 2$ for all i and $d_i = d_j = 2$ for some i and j, then since $a_i x_i^{n_i} + a_j x_j^{n_j}$ can be anything, the other $r-2$ variables are totally free and $N \simeq p^{r-1}$.

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Univariate Polynomials and Roots of Unity

Multivariate polynomials over $\mathbb{F}_p$ are bad, but univariate polynomials over $\mathbb{F}_p$ are terrible!

- Consider $x^p - x$ over $\mathbb{F}_p$. By Fermat's Little Theorem, $x^p = x$, so every element of the field is a root.
- Also, $x^p - x + 1$ has no roots. These roots clearly do not behave well.

But, hope is not lost! Univariate Polynomials are very well-understood over the roots of unity.

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Cheng 2007

We now have a deterministic (nonrandomized), polynomial time algorithm for deciding if the nth primitive root of unity ω_n satisfies $\sum_{i=1}^{k} c_i \omega_n^{e_i} = 0$, where $c_i \in \mathbb{Z}$.

- Previously, only randomized algorithms were known.
- He found a way to churn down lengthy polynomials with roots of unity having huge order into smaller ones and then using previously known techniques to do the rest.

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Essentially, roots of unity ζ_i satisfying $\sum_{i=0}^{k-1} a_i\zeta_i \equiv 0 \pmod{p}$ are no more complicated than those satisfying $\sum_{i=0}^{k-1} a_i\zeta_i = 0$, where $a_i \in \mathbb{Q}$.

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Acknowledgements

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