The distribution of short orbits of singular moduli

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Outline

The j-function

Definition

The j-function is defined by

$$j(z) = \frac{(1 + 240\sum_{n=1}^{\infty}\sum_{m|n} m^3 q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}, \quad q := e(z) = e^{2\pi i z}.$$

It has the the Fourier expansion

$$j(z) = \sum_{m=0}^{1} a(-m)q^{-m} + \sum_{m=1}^{\infty} a(m)q^m$$

where $a(-1) = 1$, and $a(0) = 744$.

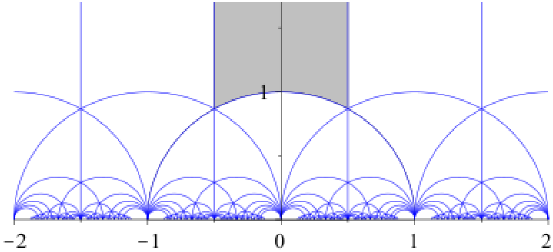
Modular functions

$j(z)$ is a modular function for $SL_2(\mathbb{Z})$, that is:

- j is meromorphic on $\mathbb{H}$, or complex differentiable on $\mathbb{H}$ except for an isolated set of points.
- j is invariant under precomposition by $SL_2(\mathbb{Z})$. So, for $\gamma \in SL_2(\mathbb{Z})$, $j(\gamma(z)) = j(z)$.

Fundamental domain of $SL_2(\mathbb{Z})$

- The fundamental domain of $SL_2(\mathbb{Z})$ acting on $\mathbb{H}$ is the region of $\mathbb{H}$ that contains exactly one point in each orbit of each element of $\mathbb{H}$. The canonical fundamental domain $\mathcal{F}$ is shaded here.



Quadratic forms

Definition

A primitive positive definite integral binary quadratic form is $Q(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$, $a > 0$, $\gcd(a, b, c) = 1$.

- Let $d = b^2 - 4ac < 0$ be the **discriminant** of Q.
- The **root** of $Q(x, 1)$ in $\mathbb{H}$ is $\tau_{[Q]} = \frac{-b+\sqrt{d}}{2a}$.

Definition

Let Q_d be the set of primitive positive definite integral binary quadratic forms of discriminant $d < 0$.

Action of $SL_2(\mathbb{Z})$ on Q_d

- The group $SL_2(\mathbb{Z})$ acts on Q_d by substitution, that is if $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, then $Q(x, y) \circ \gamma = Q(px + qy, rx + sy)$.
- We can form the quotient $G_d = Q_d / SL_2(\mathbb{Z})$.
- Gauss showed that G_d is a finite group of order $h(d)$ called the class group of d.
- By Siegel, we know that $h(d) \to \infty$ as $|d| \to \infty$.
- Let $G_d = \{[Q_1], \ldots, [Q_{h(d)}]\}$.
- Define $\mathcal{Q}_d^{\mathrm{red}} = \{Q_1, \cdots, Q_{h(d)}\}$ to be a complete set of reduced forms.

Heegner points

Definition

Let $\Lambda_d = \{\tau_{[Q_1]}, \ldots, \tau_{[Q_{h(d)}]}\}$ be the roots associated with the class representatives chosen earlier. These are called **Heegner points**.

- G_d has a simple transitive group action on Λ_d denoted by τ^σ for $\sigma \in G_d$, $\tau \in \Lambda_d$.
- This means that for any $\tau \in \Lambda_d$, $G_d\tau = \Lambda_d$.

Singular moduli

Definition

Let S_d be the set of complex numbers $\{j(\tau_{Q_1}), \ldots, j(\tau_{Q_{h(d)}})\}$. These are called **singular moduli**.

- Singular moduli are algebraic numbers, which means they are the root of some polynomial with rational coefficients.

Group characters

Definition

A *character* of a finite abelian group G is a homomorphism $\chi : G \to S^1$, the complex unit circle.

Definition

The *dual group* or *character group* of G is the group of characters of G under pointwise multiplication, written $\widehat{G}$.

Definition

Let $H < G$ be a subgroup of a finite abelian group G. Then $H^\perp := \{\chi \in \widehat{G} : \chi|_H = 1\}$ be the group of characters of G that restrict to 1 on H.

Example: Character Table of C_4

C_4, the cyclic group of order 4, has this character table:

	1	a	a^2	a^3
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	i	-1	-i
χ_3	1	-i	-1	i

You can see that $\chi(x)\chi(y) = \chi(xy)$. Note that the sum of every row besides the trivial χ_0 is 0. This is true for every character group.

Big O notation

Definition

Given two functions $f(x)$ and $g(x)$, we write $f(x) = O_\epsilon(g(x))$ if $|f(x)| \leq Cg(x)$ for some constant $C > 0$ depending only on ϵ.

Traces of singular moduli

Fix some $\tau \in \Lambda_d$.

Definition

The trace of a modular function f for some d is

$$\mathrm{Tr}_d(f) = \sum_{\sigma \in G_d} f(\tau^\sigma).$$

- $\mathrm{Tr}_d(j)$ is an algebraic integer, which means it is the root of some *monic* polynomial with integer coefficients.

A theorem of Duke

Theorem (Duke, 2006)

$$\frac{1}{h(d)}\left(\mathrm{Tr}_d(j) - \sum_{\sigma \in G_d, \mathrm{Im}(\tau^\sigma)>1} e(-\tau^\sigma)\right) \to 720$$

as $|d| \to \infty$ through $d \equiv 0, 1 \pmod 4$.

Averages of sub-orbits of singular moduli

Definition

For a subgroup $H_d < G_d$, the average of the H_d-orbit is

$$\mathrm{Av}_{H_d}(j,\tau) = \frac{1}{|H_d|}\sum_{\sigma\in H_d} j(\tau^{\sigma}).$$

Goal

We want to study the distribution of $\mathrm{Av}_{H_d}(j)$ as $|d| \to \infty$.

Main Result

Theorem

Given a subgroup $H_d < G_d$ and a Heegner point $\tau = \tau_{[Q_\tau]} \in \Lambda_d$ there exists $0 < \delta < 1/2$ such that

$$\mathrm{Av}_{H_d}(j,\tau) = M(\overline{\chi}, d, \tau) + 720 + O_\epsilon\left(|H_d|^{-1}|d|^{\delta+\epsilon}\right)$$

as $|d| \to \infty$ where

$$M(\chi, d, \tau) := \frac{\chi([Q_\tau])^{-1}}{h(d)} \sum_{m=0}^{1} \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + |d|^{\delta - \frac{1}{2}}}} C_d(Q) a(-m) e\left(-m\tau_Q\right)$$

and $C_d(Q) := \sum_{\chi \in H_d^\perp} \overline{\chi}(Q)$.

The exponent δ

Remark

Assuming the Lindelöf hypothesis for various L-functions, we can take $\delta = 9/20$.

Corollary: Good sequences of subgroups H_d

Corollary

Let $A > \delta$. If H_d satisfies $|H_d| \geq |d|^A$, then

$$\mathrm{Av}_{H_d}(j, \tau) - M(\chi, d, \tau) \to 720$$

as $|d| \to \infty$.

By Siegel's theorem, $|G_d| \gg_\epsilon |d|^{1/2-\epsilon}$.

Poisson summation

Let $H < G$ be a subgroup of a finite abelian group G.

Let $H^{\perp} := \{\chi \in \widehat{G} : \chi|_H = 1\}$ be the group of characters of G that restrict to 1 on H.

The *Poisson summation formula* states that for $f : G \to \mathbb{C}$,

$$\frac{1}{|H|}\sum_{h\in H} f(h) = \frac{1}{|G|}\sum_{\chi\in H^{\perp}}\sum_{g\in G} f(g)\overline{\chi}(g).$$

Use of Poisson Summation

By applying the Poisson summation formula to j, we can get

$$\frac{1}{|H_d|}\sum_{\sigma\in H_d} j(\tau^\sigma) = \frac{1}{h(d)}\sum_{\chi\in H_d^\perp}\sum_{\sigma\in G_d}\overline{\chi}(\sigma)j(\tau^\sigma).$$

Note that the left hand side of this equation is $\mathrm{Av}_{H_d}(j)$.

Twisted Traces

We call

$$\mathrm{Tr}_{\chi,d}(j,\tau) := \sum_{\sigma \in G_d} \chi(\sigma) j(\tau^{\sigma})$$

a *twisted trace*. So we have that

$$\mathrm{Av}_{H_d}(j,\tau) = \frac{1}{h(d)} \sum_{\chi \in H_d^{\perp}} \mathrm{Tr}_{\overline{\chi},d}(j,\tau)$$

We will first focus on analysing the twisted trace.

Background

Definition

Given two $\mathrm{SL}_2(\mathbb{Z})$-invariant functions $\phi_1, \phi_2 : \mathbb{H} \to \mathbb{C}$, we define the Petersson inner product by

$$\langle \phi_1, \phi_2 \rangle := \int_{\mathcal{F}} \phi_1(z)\overline{\phi_2}(z)\frac{dxdy}{y^2}$$

The corresponding L_2-norm is given by $||\phi||_2 := \sqrt{\langle \phi, \phi \rangle}$.

Definition

Let $\mathcal{D}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ be the space of $\mathrm{SL}_2(\mathbb{Z})$-invariant functions $\phi : \mathbb{H} \to \mathbb{C}$ such that ϕ and $\Delta\phi$ are both smooth and bounded, where $\Delta := -y^2(\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian.

For $A \in \mathbb{Z}^+$ we let Δ^A denote the composition of Δ with itself A-times.

Asymptotics for the Twisted Trace

Proposition

If $\phi \in \mathcal{D}(\mathrm{SL}_2(\mathbb{Z})/\mathbb{H})$, there is an absolute constant $0 < \delta' < 1/2$ such that

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) = C(\chi,d)\frac{3}{\pi}\langle\phi,1\rangle + O_\epsilon\left(||\Delta^A\phi||_2\,|d|^{-\delta'+\epsilon}\right)$$

for all sufficiently large $A \in \mathbb{Z}^+$ where

$$C(\chi,d) := \frac{1}{h(d)}\sum_{\sigma\in G_d}\chi(\sigma).$$

Regularizing the j function

Let $1 > \eta > 0$. Define

$$j_\eta(z) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} g_\eta(\gamma z)$$

where

$$g_\eta(z) := \sum_{m=0}^{1} a(-m)\psi_{m,\eta}(\mathrm{Im}(z))e(-mz),$$

$$\psi_{m,\eta}(y) := \phi_0\left(\frac{y - \frac{2}{\sqrt{3}}}{\eta}\right)$$

and $\phi_0(t)$ is a C^∞ function with

$$\phi_0(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

Regularizing the j function

It can be shown that for

$$j_\eta^{\text{reg}} := j - j_\eta,$$

$j_\eta^{\text{reg}} \in \mathcal{D}(\mathrm{SL}_2(\mathbb{Z})/\mathbb{H})$. This means we can apply the earlier proposition.

Decomposing the trace

The twisted trace is linear, so:

$$\mathrm{Tr}_{\chi,d}(j,\tau) = \mathrm{Tr}_{\chi,d}(j_\eta^{\mathrm{reg}},\tau) + \mathrm{Tr}_{\chi,d}(j_\eta,\tau).$$

Then by the proposition,

$$\mathrm{Tr}_{\chi,d}(j,\tau) = \mathrm{Tr}_{\chi,d}(j_\eta,\tau) + C(\chi,d)\frac{3}{\pi}\langle j_\eta^{\mathrm{reg}},1\rangle + O_\epsilon\left(||\Delta^A j_\eta^{\mathrm{reg}}||_2\,|d|^{-\delta'+\epsilon}\right).$$

We can directly calculate that $\frac{3}{\pi}\langle j_\eta^{\mathrm{reg}},1\rangle = 720$.

Dependence on choice of Heegner point

Lemma

Let $\tau = \tau_{[Q_\tau]} \in \Lambda_d$. *Then*

$$\mathrm{Tr}_{\chi,d}(j_\eta, \tau) = \chi([Q_\tau])^{-1} \sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} \chi(Q) j_\eta(\tau_Q).$$

Further decomposing $\mathrm{Tr}_{\chi,d}(j_\eta,\tau)$

We can further decompose

$$\sum_{Q\in\mathcal{Q}_d^{\mathrm{red}}} \chi(Q) j_\eta(\tau_Q) = \mathrm{T}_{\chi,d,1} + \mathrm{T}_{\chi,d,2}$$

where

$$\mathrm{T}_{\chi,d,1} := \sum_{\substack{Q\in\mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}}+\eta}} \chi(Q) \sum_{m=0}^{1} a(-m) e\left(-m\tau_Q\right)$$

$$\mathrm{T}_{\chi,d,2} := \sum_{\substack{Q\in\mathcal{Q}_d^{\mathrm{red}} \\ \frac{\sqrt{2}}{3} < y_Q \leq \frac{2}{\sqrt{3}}+\eta}} \chi(Q) \sum_{m=0}^{1} a(-m) e\left(-m\tau_Q\right).$$

More bounds

Lemma

There is a $0 < \delta'' < 1/2$ such that

$$\mathrm{T}_{\chi,d,2} = O(\eta h(d)) + O_\epsilon(\eta^{-A'}|d|^{\delta''+\epsilon}).$$

for all sufficiently large $A' \in \mathbb{Z}^+$.

Lemma

We have

$$||\Delta^A j_\eta^{\mathrm{reg}}||_2 \ll \eta^{-2A}$$

for all $A \in \mathbb{Z}^+$.

Putting it all together

Using the previous facts, we have

$$\mathrm{Tr}_{\chi,d}(j,\tau) = C(\chi,d)720 + \chi([Q_\tau])^{-1}\mathrm{T}_{\chi,d,1} + O_\epsilon(\eta^{-2A}|d|^{\delta'+\epsilon}) + O(\eta h(d)) + O_\epsilon(\eta^{-A'}|d|^{\delta''+\epsilon}).$$

Calculating the average

Using the upper bound

$$h(d) \ll |d|^{1/2+\epsilon}$$

and orthogonality of characters we get

$$\begin{aligned}\mathrm{Av}_{H_d}(j,\tau) &= \frac{1}{h(d)} \sum_{\chi \in H_d^{\perp}} \mathrm{Tr}_{\overline{\chi},d}(j,\tau) \\ &= M(\overline{\chi}, d, \tau) + 720 + O_\epsilon(|H_d|^{-1}\eta^{-2A}|d|^{\delta'+\epsilon}) \\ &\quad + O_\epsilon(|H_d|^{-1}\eta\,|d|^{1/2+\epsilon}) \quad + O_\epsilon(|H_d|^{-1}\eta^{-A'}|d|^{\delta''+\epsilon}).\end{aligned}$$

Final Optimizations

Choosing $\eta = |d|^{-b}$ and optimizing b appropriately, we get that there exists $0 < \delta < 1/2$ such that

$$\mathrm{Av}_{H_d}(j, \tau) = M(\overline{\chi}, d, \tau) + 720 + O_\epsilon\left(|H_d|^{-1}|d|^{\delta+\epsilon}\right)$$

as $|d| \to \infty$.

Next Steps

- As L-function bounds are improved, a numerical value of δ can be found and improved.
- Investigate sequences of H_d as $|d| \to \infty$ that may have combinatorial significance.
- Look into certain choices of χ which allow $\mathrm{Tr}_{\chi,d}(j,\tau)$ to describe Fourier coefficients for modular forms.

Thank you

Thank you for your time, and thank you to Dr. Masri and the organizers of this REU!