RANDOM INTERACTING PARTICLE SYSTEMS AND $U(sp_{2n})$

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Abstract. We use algebraic techniques from [1] and apply them to the Symplectic Plancherel growth model [2]. We construct central elements of $U(sp_{2n})$, calculate their states, and provide a covariance calculation.

Contents

1. Introduction

Plancherel growth process models are certain continuous dynamics for Gelfand-Tsetlin patterns that can help us better understand probabilistic phenomena in systems involving short-range dependence and simple interactions [2]. Different growth processes are constructed from the representation theory of different algebras, which affects the evolution of the growth model through factors such as its interlacing property.

This paper will study the Symplectic Plancherel growth model described in [2] by utilizing algebraic techniques from [1]. In section 2, we review the Symplectic Plancherel growth model from [2], state relevant techniques from [1], and other relevant information. In section 3 we state our main results. In section 4 we provide helpful details on how we arrived to those results.

2. Background Information

Definition 2.1. Plancherel Growth Model

The Symplectic Plancherel growth model is a Markov process on the lattice $Z_{\geq 0} \times Z_+$. Level $n \in Z_+$ has $r_n = \lfloor \frac{n+1}{2} \rfloor$ particles, which are denoted as $x_i^{(n)}$, for $i \in \{1, \cdots, r_n\}$. Each particle performs independent simple random walks, but particles are "blocked" or "pushed" according to the model's interlacing property: $x_{i+1}^{(n+1)} < x_i^{(n)} \leq x_i^{(n+1)}$ for odd n, and $x_{i+1}^{(n+1)} \leq x_i^{(n)} < x_i^{(n+1)}$ for even n.

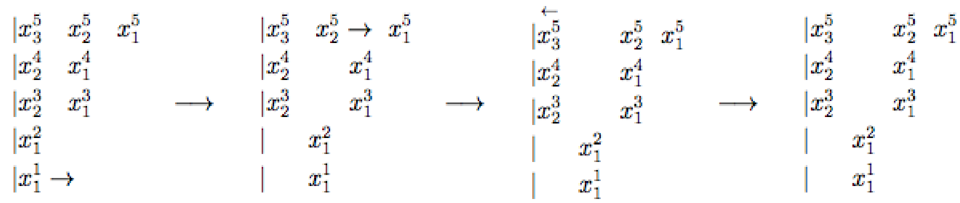

Figure 1. [2] A demonstration of how the first five levels evolve. First, a unit jump to the right by x_1^1, pushing forward several particles. Second, a unit jump to the right by x_2^2, which does not affect the first five levels. Then a unit jump to the left by x_3^5 causes it to be reflected back by the boundary.

As this model is constructed from the representation theory of sp_{2n}, we provide background information on sp_{2n} and $U(sp_{2n})$

Definition 2.2. sp_{2n} is the Lie algebra with elements $\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A = -D^T, B = B^T, C = C^T \right\}$ where A, B, C, D are $n \times n$ matrices, equipped with Lie bracket $[X, Y] = XY - YX$.

Let $i, j \in \{\pm 1, \cdots, \pm n\}$. Let E_{ij} denote the matrix with 1 in the (ij)-th entry and 0 everywhere else. Then $F_{ij} = E_{ij} - sgn(ij)E_{-j,-i}$ generate sp_{2n}.

Definition 2.3. The universal enveloping algebra of sp_{2n} is $U(sp_{2n}) = T(sp_{2n})/\langle X \otimes Y - Y \otimes X - [X, Y]\rangle$, where $T(sp_{2n}) = \mathbb{F} \oplus sp_{2n} \oplus sp_{2n}^{\otimes 2} \oplus sp_{2n}^{\otimes 3} \oplus \cdots$ is the tensor algebra of sp_{2n}.

The center $Z(U(sp_{2n}))$ is comprised of all elements that commute with all of $U(sp_{2n})$, and is generated by the central elements Φ_{2k}, for $k \in Z^+$ [1].

Construction 2.4. We now provide the steps for constructing central elements $\Phi_{2k} \in Z(U(sp_{2n}))$, as detailed in [1].

Fix a path length $k \in Z^+$. Let M denote a $d \times d$ matrix with entries being elements of an algebra A. Let the integers $v_1 < \cdots < v_d$ index the rows and columns of M. Consider all paths of length k on a graph with vertices labeled $v_1 < \cdots < v_d$ that start at v_d and end at v_d. Associate to each path $v_d \to s_1 \to \cdots \to s_{k-1} \to v_d$ a monomial $\frac{k}{\#\text{ of returns to } v_d}(M)_{v_d, s_1} \cdots (M)_{s_{k-1}, v_d}$. Then the power sum polynomial associated to M is the sum of all monomials associated to the aforementioned paths of length k.

Let F be the $2n \times 2n$ matrix with entries $F_{ij} \in U(sp_{2n})$. Let $F^{(m)} = (F_{ij})_{-m \le i,j \le m}$ and $\widetilde{F}^{(m)} = (F_{ij}))_{-m+1 \le i,j \le m}$. Then, with path length $2k$, let $\Phi_k^{(m)}$ be the power sum polynomial associated to $F^{(m)} - m * Id$, and let $\widetilde{\Phi_k}^{(m)}$ be the power sum polynomial associated to $-\widetilde{F}^{(m)} + m * Id$.

Then $\Phi_{2k} = \sum_{m=1}^n \Phi_k^{(m)} + \widetilde{\Phi_k}^{(m)}$.

We now review the techniques used in [1] that we will be applying to the Symplectic model.

Definition 2.5. For $X \in U(sp_{2n})$, define $\langle X \rangle_t = \sum_\lambda P^{(t)}(\lambda) \frac{Tr|_{V_\lambda}(X)}{\dim \lambda}$, where λ parameterizes the irreducible representations of sp_{2n}.

Remark 2.6. If $X \in Z(U(sp_{2n}))$, then X acts as $c_\lambda * Id$ on V_λ for some constant c_λ. Then $\langle X \rangle_t = \mathbb{E}[c_\lambda(\lambda(t))]$. While this nontrivial statement is not proved in this paper, the proof should follow similar arguments found in section 2,3, and 4 of [1].

Definition 2.7. $Q_t : U(sp_{2n}) \to U(sp_{2n}) \otimes U(sp_{2n})$ is the non-commutative operator defined as $(id \otimes \langle \cdot \rangle_t) \circ \Delta$, where the coproduct $\Delta : U(sp_{2n}) \to U(sp_{2n})$ is an algebra homomorphism and defined on the generators of sp_{2n}: $\Delta(F_{ij}) = F_{ij} \otimes 1 + 1 \otimes F_{ij}$. [1]

Theorem 2.8. *For $s, t \geq 0$ and $X \in U_{sp_{2n}}$, we have that $\langle Q_t X \rangle_s = \langle X \rangle_{s+t}$. [1]*

3. **Results**

We now employ several of the aforementioned facts to provide calculations of the Symplectic model that may provide clarity and confidence to the work of others.

Example 3.1. $\Phi_2 = \sum_{m=1}^{n} 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2F_{m,-m}F_{-m,m} + 4\sum_{j=-m+1}^{m-1} F_{mj}F_{jm}$
See Remark 4.1 for more details.

Example 3.2. $\Phi_4 = \sum_{m=1}^{n} \Phi_4^{(m)} + \widetilde{\Phi_4}^{(m)}$, where $\Phi_4^{(m)}$ is the sum of:

- $(F_{mm} - m)^4$
- $\sum_{j=-m}^{m-1} \frac{4}{3}F_{mj}F_{jm}(F_{mm} - m)^2 + \frac{4}{3}(F_{mm} - m)F_{mj}F_{jm}(F_{mm} - m) + \frac{4}{3}(F_{mm} - m)^2 F_{mj}F_{jm}$
- for $j \neq k$, $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} 2(F_{mm} - m)F_{mj}F_{jk}F_{km}$
- $\sum_{j=-m}^{m-1} 2(F_{mm} - m)F_{mj}(F_{jj} - m)F_{jm}$
- $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} 2F_{mj}F_{jm}F_{mk}F_{km}$
- for $j \neq k$, $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} 2F_{mj}F_{jk}F_{km}(F_{mm} - m)$
- $\sum_{j=-m}^{m-1} 2F_{mj}(F_{jj} - m)F_{jm}(F_{mm} - m)$
- for $j \neq k \neq l$, $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} \sum_{l=-m}^{m-1} 4F_{mj}F_{jk}F_{kl}F_{lm}$
- for $j \neq l$, $\sum_{j=-m}^{m-1} \sum_{l=-m}^{m-1} 4F_{mj}(F_{jj} - m)F_{jl}F_{lm}$
- for $j \neq k$, $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} 4F_{mj}F_{jk}F_{kj}F_{jm}$
- for $j \neq k$, $\sum_{j=-m}^{m-1} \sum_{k=-m}^{m-1} 4F_{mj}F_{jk}(F_{kk} - m)F_{km}$
- $\sum_{j=-m}^{m-1} 4F_{mj}(F_{jj} - m)^2 F_{jm}$

and where $\widetilde{\Phi_4}^{(m)}$ is the sum of:

- $(-F_{mm} + m)^4$
- $\sum_{j=-m+1}^{m-1} \frac{4}{3}F_{mj}F_{jm}(-F_{mm} + m)^2 + \frac{4}{3}(-F_{mm} + m)F_{mj}F_{jm}(-F_{mm} + m) + \frac{4}{3}(-F_{mm} + m)^2 F_{mj}F_{jm}$
- for $j \neq k$, $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} 2(-F_{mm} + m)F_{mj}F_{jk}F_{km}$
- $\sum_{j=-m+1}^{m-1} 2(-F_{mm} + m)F_{mj}(-F_{jj} + m)F_{jm}$
- $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} 2F_{mj}F_{jm}F_{mk}F_{km}$
- $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} 2F_{mj}F_{jk}F_{km}(-F_{mm} + m)$
- $\sum_{j=-m+1}^{m-1} 2F_{mj}(-F_{jj} + m)F_{jm}(-F_{mm} + m)$
- for $j \neq k \neq l$, $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} \sum_{l=-m+1}^{m-1} 4F_{mj}F_{jk}F_{kl}F_{lm}$
- for $j \neq l$, $\sum_{j=-m+1}^{m-1} \sum_{l=-m+1}^{m-1} 4F_{mj}(-F_{jj} + m)F_{jl}F_{lm}$
- for $j \neq k$, $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} 4F_{mj}F_{jk}F_{kj}F_{jm}$
- for $j \neq k$, $\sum_{j=-m+1}^{m-1} \sum_{k=-m+1}^{m-1} 4F_{mj}F_{jk}(-F_{kk} + m)F_{km}$
- $\sum_{j=-m+1}^{m-1} 4F_{mj}(-F_{jj} + m)^2 F_{jm}$

See Remark 4.2 for more details.

Proposition 3.3. $\langle \Phi_2 \rangle_t = \frac{n(n+1)(2n+1)}{3} + (8n^2+4n)t$
See Remark 4.3 for more details.

Proposition 3.4. $\langle \Phi_4 \rangle_t = (64n^3+48n^2+8n)t^2 + (40n^4-24n^3+40n^2+4n)t + \frac{6n^5+15n^3+10n^3-n}{15}$
See Remark 4.4 for more details.

Proposition 3.5. *For Φ_{2k}, we can express $Q_t\Phi_{2k}$ as linear combination of $\{\Phi_{2k}\}_{k=1}^{\infty}$ [1]. In particular, we have $Q_t\Phi_4 = \Phi_4 + (16nt+4t)\Phi_2 + (64n^3+48n^2+8n)t^2 + (40n^4-24n^3+40n^2+4n)t - (16nt+4t)(\frac{n(n+1)(2n+1)}{3})$*
See Remark 4.5 for more details.

In section 5 of [1], the technique used in example 8 is used to explore the covariance structure of the model. We adapt this technique to the case of Φ_2 to obtain:

Proposition 3.6. $\lim_{L\to\infty} \langle \frac{\Phi_2^{(\eta_1 L)} - \langle \Phi_2^{(\eta_1 L)} \rangle_{\tau_1 L}}{L^2} * \frac{Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} - \langle Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} \rangle_{\tau_1 L}}{L^2} \rangle_{\tau_1 L} = (32\eta_2\eta^2 + 32\eta_1\eta^2 - 32\eta^3)\tau_1 + 64\eta^2\tau_1^2$*, where* $\eta = \min\{\eta_1, \eta_2\}$
See Remark 4.6 for more details. More information on the covariance structure of plancherel growth process models may be found in section 5 of [1].

4. **Proofs and Useful Details**

The following is a compilation of proofs and useful details that will aid readers in understanding how the results were obtained.

Remark 4.1. First we calculate $\Phi_2^{(m)}$. The path length is 2, implying that either we have the path $m \to m \to m$, giving the monomial $(F_{mm}-m)^2$ or $m \to j \to m$, giving the monomial $2F_{mj}F_{jm}$. This yields $\Phi_2^{(m)} = (F_{mm}-m)^2 + \sum_{j=-m}^{m-1} 2F_{mj}F_{jm}$. An analagous argument yields $\widetilde{\Phi_2}^{(m)} = (-F_{mm}+m)^2 + \sum_{j=-m+1}^{m-1} 2F_{mj}F_{jm}$. Note $j \neq 0$ in both summations.

Remark 4.2. The following is a more detailed casework of the construction of $\Phi_4^{(m)}$:

- if the particle returns 4 times, then we get $(F_{mm}-m)^4$
- if the particle returns 3 times, we get $\sum_{j=-m}^{m-1} \frac{4}{3}F_{mj}F_{jm}(F_{mm}-m)^2 + \frac{4}{3}(F_{mm}-m)F_{mj}F_{jm}(F_{mm}-m) + \frac{4}{3}(F_{mm}-m)^2F_{mj}F_{jm}$
- if the particle returns 2 times, for the case
  - $m \to m \to j \to k \to m$, where $j \neq k$, gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2(F_{mm}-m)F_{mj}F_{jk}F_{km}$
  - $m \to m \to j \to j \to m$ gives $\sum_{j=-m}^{m-1} 2(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$
  - $m \to j \to m \to k \to m$ gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2F_{mj}F_{jm}F_{mk}F_{km}$
  - $m \to j \to k \to m \to m$, where $j \neq k$, gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2F_{mj}F_{jk}F_{km}(F_{mm}-m)$
  - $m \to j \to j \to m \to m$ gives $\sum_{j=-m}^{m-1} 2F_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$
- if the particle only returns once, and follows the path $m \to j \to k \to l \to m$
  - and $j \neq k \neq l$, this gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\sum_{l=-m}^{m-1} 4F_{mj}F_{jk}F_{kl}F_{lm}$
  - and $j = k \neq l$, this gives $\sum_{j=-m}^{m-1}\sum_{l=-m}^{m-1} 4F_{mj}(F_{jj}-m)F_{jl}F_{lm}$
  - and $j = l \neq k$, this gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 4F_{mj}F_{jk}F_{kj}F_{jm}$
  - and $k = l \neq j$, this gives $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 4F_{mj}F_{jk}(F_{kk}-m)F_{km}$

   - and $j=k=l$, this gives $\sum_{j=-m}^{m-1} 4F_{mj}(F_{jj}-m)^2F_{jm}$

The following is a more detailed casework of the construction of $\widetilde{\Phi_4}^{(m)}$:

- if the particle returns 4 times, then we get $(-F_{mm}+m)^4$
- if the particle returns 3 times, we get $\sum_{j=-m+1}^{m-1} \frac{4}{3}F_{mj}F_{jm}(-F_{mm}+m)^2 + \frac{4}{3}(-F_{mm}+m)F_{mj}F_{jm}(-F_{mm}+m) + \frac{4}{3}(-F_{mm}+m)^2F_{mj}F_{jm}$
- if the particle returns 2 times, for the case
   - $m\to m\to j\to k\to m$, where $j\neq k$, gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 2(-F_{mm}+m)F_{mj}F_{jk}F_{km}$
   - $m\to m\to j\to j\to m$ gives $\sum_{j=-m+1}^{m-1} 2(-F_{mm}+m)F_{mj}(-F_{jj}+m)F_{jm}$
   - $m\to j\to m\to k\to m$ gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 2F_{mj}F_{jm}F_{mk}F_{km}$
   - $m\to j\to k\to m\to m$ gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 2F_{mj}F_{jk}F_{km}(-F_{mm}+m)$
   - $m\to j\to j\to m\to m$ gives $\sum_{j=-m+1}^{m-1} 2F_{mj}(-F_{jj}+m)F_{jm}(-F_{mm}+m)$
- if the particle only returns once, and follows the path $m\to j\to k\to l\to m$
   - and $j\neq k\neq l$, this gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\sum_{l=-m+1}^{m-1} 4F_{mj}F_{jk}F_{kl}F_{lm}$
   - and $j=k\neq l$, this gives $\sum_{j=-m+1}^{m-1}\sum_{l=-m+1}^{m-1} 4F_{mj}(-F_{jj}+m)F_{jl}F_{lm}$
   - and $j=l\neq k$, this gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 4F_{mj}F_{jk}F_{kj}F_{jm}$
   - and $k=l\neq j$, this gives $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 4F_{mj}F_{jk}(-F_{kk}+m)F_{km}$
   - and $j=k=l$, this gives $\sum_{j=-m+1}^{m-1} 4F_{mj}(-F_{jj}+m)^2F_{jm}$

Remark 4.3. We have that $Q_t\Phi_2 = \sum_{m=1}^{n} 2(\langle F_{mm}^2\rangle_t + F_{mm}^2) - 4mF_{mm} + 2m^2 + 2[\langle F_{m,-m}F_{-m,m}\rangle_t + F_{m,-m}F_{-m,m}\rangle_t] + 4\sum_{j=-m+1}^{m-1}\langle F_{mj}F_{jm}\rangle_t + F_{mj}F_{jm}$.

Then $\langle\Phi_2\rangle_t = \lim_{\epsilon\to 0}\langle\Phi_2\rangle_{t+\epsilon} = \lim_{\epsilon\to 0}\langle Q_t\Phi_2\rangle_\epsilon$.

Remark 4.4. Calculating $\langle\Phi_4\rangle_t$:

Note that $\langle\Phi_4\rangle_{t+\epsilon} = \langle Q_t\Phi_4\rangle_\epsilon = \langle(id\otimes\langle\cdot\rangle_t)\Delta\Phi_4\rangle_\epsilon$.

So first we calculate $\Delta\Phi_4$. Since $\Phi_4 = \sum_{m=1}^{n}\Phi_4^{(m)} + \widetilde{\Phi_4^{(m)}}$, we will calculate the coproduct for general m for each of the two Φ's.

Necessary calculations for $\Delta\Phi_4^{(m)}$:

- $\Delta(F_{mm}-m)^4$
   - $\Delta(F_{mm}-m)^4 = (\Delta(F_{mm})-\Delta(m))^4 = (F_{mm}\otimes 1 + 1\otimes F_{mm} - m\otimes 1)^4$. Note that $(F_{mm}\otimes 1 + 1\otimes F_{mm} - m\otimes 1)^2 = F_{mm}^2\otimes 1 + 1\otimes F_{mm}^2 + 2F_{mm}\otimes F_{mm} - 2mF_{mm}\otimes 1 - 1\otimes 2mF_{mm} + m^2(1\otimes 1)$. **Thus, we have that the coproduct is:** $F_{mm}^4\otimes 1 + 1\otimes F_{mm}^4 + 6F_{mm}^2\otimes F_{mm}^2 + 4F_{mm}^3\otimes F_{mm} + 4F_{mm}\otimes F_{mm}^3 - 4mF_{mm}^3\otimes 1 - 4m1\otimes F_{mm}^3 - 12mF_{mm}^2\otimes F_{mm} - 12mF_{mm}\otimes F_{mm}^2 + 6m^2F_{mm}^2\otimes 1 + 6m^21\otimes F_{mm}^2 + 12m^2F_{mm}\otimes F_{mm} - 4m^3F_{mm}\otimes 1 - 4m^31\otimes F_{mm} + m^4(1\otimes 1)$.
- $\Delta(F_{mj}F_{jm}(F_{mm}-m)^2)$
   - $= \Delta F_{mj}\Delta F_{jm}(\Delta(F_{mm}-m))^2 = (F_{mj}\otimes 1 + 1\otimes F_{mj})(F_{jm}\otimes 1 + 1\otimes F_{jm})(F_{mm}^2\otimes 1 + 1\otimes F_{mm}^2 + 2F_{mm}\otimes F_{mm} - 2mF_{mm}\otimes 1 - 2m1\otimes F_{mm} + m^2(1\otimes 1))$. **Thus, we have that the coproduct is:**

$F_{mj}F_{jm}F_{mm}^2\otimes 1+1\otimes F_{mj}F_{jm}F_{mm}^2+F_{mj}F_{jm}\otimes F_{mm}^2+F_{mm}^2\otimes F_{mj}F_{jm}+2F_{mj}F_{jm}F_{mm}\otimes F_{mm}+2F_{mm}\otimes F_{mj}F_{jm}F_{mm}-2mF_{mj}F_{jm}F_{mm}\otimes 1-2m1\otimes F_{mj}F_{jm}F_{mm}-2mF_{mj}F_{jm}\otimes F_{mm}-2mF_{mm}\otimes F_{mj}F_{jm}+m^2F_{mj}F_{jm}\otimes 1+m^2 1\otimes F_{mj}F_{jm}+m^2F_{mj}\otimes F_{jm}+m^2F_{jm}\otimes F_{mj}+F_{mj}F_{mm}^2\otimes F_{jm}+F_{jm}\otimes F_{mj}F_{mm}^2+F_{mj}\otimes F_{jm}F_{mm}^2+F_{jm}F_{mm}^2\otimes F_{mj}+2F_{mj}F_{mm}\otimes F_{jm}F_{mm}+2F_{jm}F_{mm}\otimes F_{mj}F_{mm}-2mF_{mj}F_{mm}\otimes F_{jm}-2mF_{jm}\otimes F_{mj}F_{mm}-2mF_{mj}\otimes F_{jm}F_{mm}-2mF_{jm}F_{mm}\otimes F_{mj}$

- $\Delta((F_{mm}-m)F_{mj}F_{jm}(F_{mm}-m))$
  - $=\Delta(F_{mm}-m)\Delta F_{mj}\Delta F_{jm}\Delta(F_{mm}-m)$. Note that $\Delta(F_{mm}-m)\Delta F_{mj}=F_{mm}F_{mj}\otimes 1+F_{mm}\otimes F_{mj}+F_{mj}\otimes F_{mm}+1\otimes F_{mm}F_{mj}-mF_{mj}\otimes 1-m1\otimes F_{mj}$. Also note that $\Delta F_{jm}\Delta(F_{mm}-m)=(F_{jm}\otimes 1+1\otimes F_{jm})(F_{mm}\otimes 1+1\otimes F_{mm}-m\otimes 1)$. **Thus we have the coproduct is:** $F_{mm}F_{mj}F_{jm}F_{mm}\otimes 1+1\otimes F_{mm}F_{mj}F_{jm}F_{mm}+F_{mm}F_{mj}F_{jm}\otimes F_{mm}+F_{mm}\otimes F_{mm}F_{mj}F_{jm}-mF_{mm}F_{mj}F_{jm}\otimes 1-m1\otimes F_{mm}F_{mj}F_{jm}+F_{mm}F_{mj}F_{mm}\otimes F_{jm}+F_{jm}\otimes F_{mm}F_{mj}F_{mm}+F_{mm}F_{mj}\otimes F_{jm}F_{mm}+F_{jm}F_{mm}\otimes F_{mm}F_{mj}-mF_{mm}F_{mj}\otimes F_{jm}-mF_{jm}\otimes F_{mm}F_{mj}+F_{mm}F_{jm}F_{mm}\otimes F_{mj}+F_{mj}\otimes F_{mm}F_{jm}F_{mm}+F_{mm}F_{jm}\otimes F_{mj}F_{mm}+F_{mj}F_{mm}\otimes F_{mm}F_{jm}-mF_{mm}F_{jm}\otimes F_{mj}-mF_{mj}\otimes F_{mm}F_{jm}+F_{mm}^2\otimes F_{mj}F_{jm}+F_{mj}F_{jm}\otimes F_{mm}^2+F_{mm}\otimes F_{mj}F_{jm}F_{mm}+F_{mj}F_{jm}F_{mm}\otimes F_{mm}-mF_{mm}\otimes F_{mj}F_{jm}-mF_{mj}F_{jm}\otimes F_{mm}-mF_{mj}F_{jm}F_{mm}\otimes 1-m1\otimes F_{mj}F_{jm}F_{mm}-mF_{mj}F_{jm}\otimes F_{mm}-mF_{mm}\otimes F_{mj}F_{jm}+m^2F_{mj}F_{jm}\otimes 1+m^2 1\otimes F_{mj}F_{jm}-mF_{mj}F_{mm}\otimes F_{jm}-mF_{jm}\otimes F_{mj}F_{mm}-mF_{mj}\otimes F_{jm}F_{mm}-mF_{jm}F_{mm}\otimes F_{mj}+m^2F_{mj}\otimes F_{jm}+m^2F_{jm}\otimes F_{mj}$
- $\Delta((F_{mm}-m)^2F_{mj}F_{jm})$
  - Note that this is equal to $(F_{mm}^2\otimes 1+1\otimes F_{mm}^2+2F_{mm}\otimes F_{mm}-2mF_{mm}\otimes 1-2m1\otimes F_{mm}+m^2(1\otimes 1))(F_{mj}\otimes 1+1\otimes F_{mj})(F_{jm}\otimes 1+1\otimes F_{jm})$. **Thus we have that the coproduct is:** $F_{mm}^2F_{mj}F_{jm}\otimes 1+1\otimes F_{mm}^2F_{mj}F_{jm}+F_{mj}F_{jm}\otimes F_{mm}^2+F_{mm}^2\otimes F_{mj}F_{jm}+2F_{mm}F_{mj}F_{jm}\otimes F_{mm}+2F_{mm}\otimes F_{mm}F_{mj}F_{jm}-2mF_{mm}F_{mj}F_{jm}\otimes 1-2m1\otimes F_{mm}F_{mj}F_{jm}-2mF_{mj}F_{jm}\otimes F_{mm}-2mF_{mm}\otimes F_{mj}F_{jm}+m^2F_{mj}F_{jm}\otimes 1+m^2 1\otimes F_{mj}F_{jm}+m^2F_{mj}\otimes F_{jm}+m^2F_{jm}\otimes F_{mj}+F_{mm}^2F_{mj}\otimes F_{jm}+F_{jm}\otimes F_{mm}^2F_{mj}+F_{mj}\otimes F_{mm}^2F_{jm}+F_{mm}^2F_{jm}\otimes F_{mj}+2F_{mm}F_{mj}\otimes F_{mm}F_{jm}+2F_{mm}F_{jm}\otimes F_{mm}F_{mj}-2mF_{mm}F_{mj}\otimes F_{jm}-2mF_{jm}\otimes F_{mm}F_{mj}-2mF_{mj}\otimes F_{mm}F_{jm}-2mF_{mm}F_{jm}\otimes F_{mj}$
- $\Delta((F_{mm}-m)F_{mj}F_{jk}F_{km})$
  - We have that $\Delta(F_{mm}-m)F_{mj}=F_{mm}F_{mj}\otimes 1+F_{mm}\otimes F_{mj}+F_{mj}\otimes F_{mm}+1\otimes F_{mm}F_{mj}-mF_{mj}\otimes 1-m1\otimes F_{mj}$, and $\Delta F_{jk}F_{km}=F_{jk}F_{km}\otimes 1+F_{jk}\otimes F_{km}+F_{km}\otimes F_{jk}+1\otimes F_{jk}F_{km}$. **Thus, we have the coproduct is:** $F_{mm}F_{mj}F_{jk}F_{km}\otimes 1+F_{mm}F_{mj}F_{jk}\otimes F_{km}+F_{mm}F_{mj}F_{km}\otimes F_{jk}+F_{mm}F_{mj}\otimes F_{jk}F_{km}+F_{mm}F_{jk}F_{km}\otimes F_{mj}+F_{mm}F_{jk}\otimes F_{mj}F_{km}+F_{mm}F_{km}\otimes F_{mj}F_{jk}+F_{mm}\otimes F_{mj}F_{jk}F_{km}+F_{mj}F_{jk}F_{km}\otimes F_{mm}+F_{mj}F_{jk}\otimes F_{mm}F_{km}+F_{mj}F_{km}\otimes F_{mm}F_{jk}+F_{mj}\otimes F_{mm}F_{jk}F_{km}+F_{jk}F_{km}\otimes F_{mm}F_{mj}+F_{jk}\otimes F_{mm}F_{mj}F_{km}+F_{km}\otimes F_{mm}F_{mj}F_{jk}+1\otimes F_{mm}F_{mj}F_{jk}F_{km}-mF_{mj}F_{jk}F_{km}\otimes 1+mF_{mj}F_{jk}\otimes F_{km}-mF_{mj}F_{km}\otimes F_{jk}-mF_{mj}\otimes F_{jk}F_{km}-mF_{jk}F_{km}\otimes F_{mj}-mF_{jk}\otimes F_{mj}F_{jk}-mF_{km}\otimes F_{mj}F_{jk}-m1\otimes F_{mj}F_{jk}F_{km}$
- $\Delta(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$

- We have that $\Delta(F_{mm}-m)F_{mj} = F_{mm}F_{mj}\otimes 1 + F_{mm}\otimes F_{mj} + F_{mj}\otimes F_{mm} + 1\otimes F_{mm}F_{mj} - mF_{mj}\otimes 1 - m1\otimes F_{mj}$. We also have that $\Delta(F_{jj}-m)F_{jm} = F_{jj}F_{jm}\otimes 1 + F_{jj}\otimes F_{jm} + F_{jm}\otimes F_{jj} + 1\otimes F_{jj}F_{jm} - mF_{jm}\otimes 1 - m1\otimes F_{jm}$. **Thus, we have the coproduct is:** $F_{mm}F_{mj}F_{jj}F_{jm}\otimes 1 + F_{mm}F_{mj}F_{jj}\otimes F_{jm} + F_{mm}F_{mj}F_{jm}\otimes F_{jj} + F_{mm}F_{mj}\otimes F_{jj}F_{jm} - mF_{mm}F_{mj}F_{jm}\otimes 1 - mF_{mm}F_{mj}\otimes F_{jm} + F_{mm}F_{jj}F_{jm}\otimes F_{mj} + F_{mm}F_{jj}\otimes F_{mj}F_{jm} + F_{mm}F_{jm}\otimes F_{mj}F_{jj} + F_{mm}\otimes F_{mj}F_{jj}F_{jm} - mF_{mm}F_{jm}\otimes F_{mj} - mF_{mm}\otimes F_{mj}F_{jm} + F_{mj}F_{jj}F_{jm}\otimes F_{mm} + F_{mj}F_{jj}\otimes F_{mm}F_{jm} + F_{mj}F_{jm}\otimes F_{mm}F_{jj} + F_{mj}\otimes F_{mm}F_{jj}F_{jm} - mF_{mj}F_{jm}\otimes F_{mm} - mF_{mj}\otimes F_{mm}F_{jm} + F_{jj}F_{jm}\otimes F_{mm}F_{mj} + F_{jj}\otimes F_{mm}F_{mj}F_{jm} + F_{jm}\otimes F_{mm}F_{mj}F_{jj} + 1\otimes F_{mm}F_{mj}F_{jj}F_{jm} - mF_{jm}\otimes F_{mm}F_{mj} - m1\otimes F_{mm}F_{mj}F_{jm} - mF_{mj}F_{jj}F_{jm}\otimes 1 - mF_{mj}F_{jj}\otimes F_{jm} - mF_{mj}F_{jm}\otimes F_{jj} - mF_{mj}\otimes F_{jj}F_{jm} + m^2F_{mj}F_{jm}\otimes 1 + m^2F_{mj}\otimes F_{jm} - mF_{jj}F_{jm}\otimes F_{mj} - mF_{jj}\otimes F_{mj}F_{jm} - mF_{jm}\otimes F_{mj}F_{jj} - m1\otimes F_{mj}F_{jj}F_{jm} + m^2F_{jm}\otimes F_{mj} + m^21\otimes F_{mj}F_{jm}$

- $\Delta F_{mj}F_{jm}F_{mk}F_{km}$
  - We have $\Delta F_{mj}F_{jm} = F_{mj}F_{jm}\otimes 1 + 1\otimes F_{mj}F_{jm} + F_{mj}\otimes F_{jm} + F_{jm}\otimes F_{mj}$, and we also have $\Delta F_{mk}F_{km} = F_{mk}F_{km}\otimes 1 + 1\otimes F_{mk}F_{km} + F_{mk}\otimes F_{km} + F_{km}\otimes F_{mk}$. **Thus we have the coproduct is:** $F_{mj}F_{jm}F_{mk}F_{km}\otimes 1 + F_{mj}F_{jm}\otimes F_{mk}F_{km} + F_{mj}F_{jm}F_{mk}\otimes F_{km} + F_{mj}F_{jm}F_{km}\otimes F_{mk} + F_{mk}F_{km}\otimes F_{mj}F_{jm} + 1\otimes F_{mj}F_{jm}F_{mk}F_{km} + F_{mk}\otimes F_{mj}F_{jm}F_{km} + F_{km}\otimes F_{mj}F_{jm}F_{mk} + F_{mj}F_{mk}F_{km}\otimes F_{jm} + F_{mj}\otimes F_{jm}F_{mk}F_{km} + F_{mj}F_{mk}\otimes F_{jm}F_{km} + F_{mj}F_{km}\otimes F_{jm}F_{mk} + F_{jm}F_{mk}F_{km}\otimes F_{mj} + F_{jm}\otimes F_{mj}F_{mk}F_{km} + F_{jm}F_{mk}\otimes F_{mj}F_{km} + F_{jm}F_{km}\otimes F_{mj}F_{mk}$
- $\Delta F_{mj}F_{jk}F_{km}(F_{mm}-m)$
  - We have that $\Delta F_{mj}F_{jk} = F_{mj}F_{jk}\otimes 1 + 1\otimes F_{mj}F_{jk} + F_{mj}\otimes F_{jk} + F_{jk}\otimes F_{mj}$. We also have that $\Delta F_{km}(F_{mm}-m) = F_{km}F_{mm}\otimes 1 + F_{km}\otimes F_{mm} - mF_{km}\otimes 1 + F_{mm}\otimes F_{km} + 1\otimes F_{km}F_{mm} - m1\otimes F_{km}$. **Thus, we have the coproduct is:** $F_{mj}F_{jk}F_{km}F_{mm}\otimes 1 + F_{mj}F_{jk}F_{km}\otimes F_{mm} - mF_{mj}F_{jk}F_{km}\otimes 1 + F_{mj}F_{jk}F_{mm}\otimes F_{km} + F_{mj}F_{jk}\otimes F_{km}F_{mm} - mF_{mj}F_{jk}\otimes F_{km} + F_{km}F_{mm}\otimes F_{mj}F_{jk} + F_{km}\otimes F_{mj}F_{jk}F_{mm} - mF_{km}\otimes F_{mj}F_{jk} + F_{mm}\otimes F_{mj}F_{jk}F_{km} + 1\otimes F_{mj}F_{jk}F_{km}F_{mm} - m1\otimes F_{mj}F_{jk}F_{km} + F_{mj}F_{km}F_{mm}\otimes F_{jk} + F_{mj}F_{km}\otimes F_{jk}F_{mm} - mF_{mj}F_{km}\otimes F_{jk} + F_{mj}F_{mm}\otimes F_{jk}F_{km} + F_{mj}\otimes F_{jk}F_{km}F_{mm} - mF_{mj}\otimes F_{jk}F_{km} + F_{jk}F_{km}F_{mm}\otimes F_{mj} + F_{jk}F_{km}\otimes F_{mj}F_{mm} - mF_{jk}F_{km}\otimes F_{mj} + F_{jk}F_{mm}\otimes F_{mj}F_{km} + F_{jk}\otimes F_{mj}F_{km}F_{mm} - mF_{jk}\otimes F_{mj}F_{km}$
- $\Delta F_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$
  - We have that $\Delta F_{mj}(F_{jj}-m) = F_{mj}F_{jj}\otimes 1 + F_{mj}\otimes F_{jj} - mF_{mj}\otimes 1 + F_{jj}\otimes F_{mj} + 1\otimes F_{mj}F_{jj} - m1\otimes F_{mj}$. We have that $\Delta F_{jm}(F_{mm}-m) = F_{jm}F_{mm}\otimes 1 + F_{jm}\otimes F_{mm} - mF_{jm}\otimes 1 + F_{mm}\otimes F_{jm} + 1\otimes F_{jm}F_{mm} - m1\otimes F_{jm}$. **Thus, we have the coproduct is:** $F_{mj}F_{jj}F_{jm}F_{mm}\otimes 1 + F_{mj}F_{jj}F_{jm}\otimes F_{mm} - mF_{mj}F_{jj}F_{jm}\otimes 1 + F_{mj}F_{jj}F_{mm}\otimes F_{jm} + F_{mj}F_{jj}\otimes F_{jm}F_{mm} - mF_{mj}F_{jj}\otimes F_{jm} + F_{mj}F_{jm}F_{mm}\otimes F_{jj} + F_{mj}F_{jm}\otimes F_{jj}F_{mm} - mF_{mj}F_{jm}\otimes F_{jj} + F_{mj}F_{mm}\otimes F_{jj}F_{jm} + F_{mj}\otimes F_{jj}F_{jm}F_{mm} - mF_{mj}\otimes F_{jj}F_{jm} - mF_{mj}F_{jm}F_{mm}\otimes 1 - mF_{mj}F_{jm}\otimes F_{mm} + m^2F_{mj}F_{jm}\otimes 1 - mF_{mj}F_{mm}\otimes F_{jm} - mF_{mj}\otimes F_{jm}F_{mm} + m^2F_{mj}\otimes F_{jm} + F_{jj}F_{jm}F_{mm}\otimes F_{mj} + F_{jj}F_{jm}\otimes F_{mj}F_{mm} - mF_{jj}F_{jm}\otimes F_{mj} + F_{jj}F_{mm}\otimes F_{mj}F_{jm} + F_{jj}\otimes F_{mj}F_{jm}F_{mm} - mF_{jj}\otimes F_{mj}F_{jm} + F_{jm}F_{mm}\otimes F_{mj}F_{jj} + F_{jm}\otimes$

$F_{mj}F_{jj}F_{mm} - mF_{jm} \otimes F_{mj}F_{jj} + F_{mm} \otimes F_{mj}F_{jj}F_{jm} + 1 \otimes F_{mj}F_{jj}F_{jm}F_{mm} - m1 \otimes F_{mj}F_{jj}F_{jm} - mF_{jm}F_{mm} \otimes F_{mj} - mF_{jm} \otimes F_{mj}F_{mm} + m^2F_{jm} \otimes F_{mj} - mF_{mm} \otimes F_{mj}F_{jm} - m1 \otimes F_{mj}F_{jm}F_{mm} + m^21 \otimes F_{mj}F_{jm}$

- $\Delta F_{mj}F_{jk}F_{kl}F_{lm}$
  - We have that $\Delta F_{mj}F_{jk} = F_{mj}F_{jk} \otimes 1 + 1 \otimes F_{mj}F_{jk} + F_{mj} \otimes F_{jk} + F_{jk} \otimes F_{mj}$. We also have that $\Delta F_{kl}F_{lm} = F_{kl}F_{lm} \otimes 1 + 1 \otimes F_{kl}F_{lm} + F_{kl} \otimes F_{lm} + F_{lm} \otimes F_{kl}$. **Thus, the coproduct is:** $F_{mj}F_{jk}F_{kl}F_{lm} \otimes 1 + F_{mj}F_{jk} \otimes F_{kl}F_{lm} + F_{mj}F_{jk}F_{kl} \otimes F_{lm} + F_{mj}F_{jk}F_{lm} \otimes F_{kl} + F_{kl}F_{lm} \otimes F_{mj}F_{jk} + 1 \otimes F_{mj}F_{jk}F_{kl}F_{lm} + F_{kl} \otimes F_{mj}F_{jk}F_{lm} + F_{lm} \otimes F_{mj}F_{jk}F_{kl} + F_{mj}F_{kl}F_{lm} \otimes F_{jk} + F_{mj} \otimes F_{jk}F_{kl}F_{lm} + F_{mj}F_{kl} \otimes F_{jk}F_{lm} + F_{mj}F_{lm} \otimes F_{jk}F_{kl} + F_{jk}F_{kl}F_{lm} \otimes F_{mj} + F_{jk} \otimes F_{mj}F_{kl}F_{lm} + F_{jk}F_{kl} \otimes F_{mj}F_{lm} + F_{jk}F_{lm} \otimes F_{mj}F_{kl}$
- $\Delta F_{mj}(F_{jj} - m)F_{jl}F_{lm}$
  - We have that $\Delta F_{mj}(F_{jj} - m) = F_{mj}F_{jj} \otimes 1 + F_{mj} \otimes F_{jj} - mF_{mj} \otimes 1 + F_{jj} \otimes F_{mj} + 1 \otimes F_{mj}F_{jj} - m1 \otimes F_{mj}$. We have that $\Delta F_{jl}F_{lm} = F_{jl}F_{lm} \otimes 1 + 1 \otimes F_{jl}F_{lm} + F_{jl} \otimes F_{lm} + F_{lm} \otimes F_{jl}$. **Thus we have the coproduct is:** $F_{mj}F_{jj}F_{jl}F_{lm} \otimes 1 + F_{mj}F_{jj} \otimes F_{jl}F_{lm} + F_{mj}F_{jj}F_{jl} \otimes F_{lm} + F_{mj}F_{jj}F_{lm} \otimes F_{jl} + F_{mj}F_{jl}F_{lm} \otimes F_{jj} + F_{mj} \otimes F_{jj}F_{jl}F_{lm} + F_{mj}F_{jl} \otimes F_{jj}F_{lm} + F_{mj}F_{lm} \otimes F_{jj}F_{jl} - mF_{mj}F_{jl}F_{lm} \otimes 1 - mF_{mj} \otimes F_{jl}F_{lm} - mF_{mj}F_{jl} \otimes F_{lm} - mF_{mj}F_{lm} \otimes F_{jl} + F_{jj}F_{jl}F_{lm} \otimes F_{mj} + F_{jj} \otimes F_{mj}F_{jl}F_{lm} + F_{jj}F_{jl} \otimes F_{mj}F_{lm} + F_{jj}F_{lm} \otimes F_{mj}F_{jl} + F_{jl}F_{lm} \otimes F_{mj}F_{jj} + 1 \otimes F_{mj}F_{jj}F_{jl}F_{lm} + F_{jl} \otimes F_{mj}F_{jj}F_{lm} + F_{lm} \otimes F_{mj}F_{jj}F_{jl} - mF_{jl}F_{lm} \otimes F_{mj} - m1 \otimes F_{mj}F_{jl}F_{lm} - mF_{jl} \otimes F_{mj}F_{lm} - mF_{lm} \otimes F_{mj}F_{jl}$
- $\Delta F_{mj}F_{jk}F_{kj}F_{jm}$
  - We have that $\Delta F_{mj}F_{jk} = F_{mj}F_{jk} \otimes 1 + 1 \otimes F_{mj}F_{jk} + F_{mj} \otimes F_{jk} + F_{jk} \otimes F_{mj}$. We also have that $\Delta F_{kj}F_{jm} = F_{kj}F_{jm} \otimes 1 + 1 \otimes F_{kj}F_{jm} + F_{kj} \otimes F_{jm} + F_{jm} \otimes F_{kj}$. **Thus, we have the coproduct is:** $F_{mj}F_{jk}F_{kj}F_{jm} \otimes 1 + F_{mj}F_{jk} \otimes F_{kj}F_{jm} + F_{mj}F_{jk}F_{kj} \otimes F_{jm} + F_{mj}F_{jk}F_{jm} \otimes F_{kj} + F_{kj}F_{jm} \otimes F_{mj}F_{jk} + 1 \otimes F_{mj}F_{jk}F_{kj}F_{jm} + F_{kj} \otimes F_{mj}F_{jk}F_{jm} + F_{jm} \otimes F_{mj}F_{jk}F_{kj} + F_{mj}F_{kj}F_{jm} \otimes F_{jk} + F_{mj} \otimes F_{jk}F_{kj}F_{jm} + F_{mj}F_{kj} \otimes F_{jk}F_{jm} + F_{mj}F_{jm} \otimes F_{jk}F_{kj} + F_{jk}F_{kj}F_{jm} \otimes F_{mj} + F_{jk} \otimes F_{mj}F_{kj}F_{jm} + F_{jk}F_{kj} \otimes F_{mj}F_{jm} + F_{jk}F_{jm} \otimes F_{mj}F_{kj}$
- $\Delta F_{mj}F_{jk}(F_{kk} - m)F_{km}$
  - We have that $\Delta F_{mj}F_{jk} = F_{mj}F_{jk} \otimes 1 + 1 \otimes F_{mj}F_{jk} + F_{mj} \otimes F_{jk} + F_{jk} \otimes F_{mj}$. We also have that $\Delta(F_{kk} - m)F_{km} = F_{kk}F_{km} \otimes 1 + F_{kk} \otimes F_{km} + F_{km} \otimes F_{kk} + 1 \otimes F_{kk}F_{km} - mF_{km} \otimes 1 - m1 \otimes F_{km}$. **Thus, we have the coproduct is:** $F_{mj}F_{jk}F_{kk}F_{km} \otimes 1 + F_{mj}F_{jk}F_{kk} \otimes F_{km} + F_{mj}F_{jk}F_{km} \otimes F_{kk} + F_{mj}F_{jk} \otimes F_{kk}F_{km} - mF_{mj}F_{jk}F_{km} \otimes 1 - mF_{mj}F_{jk} \otimes F_{km} + F_{kk}F_{km} \otimes F_{mj}F_{jk} + F_{kk} \otimes F_{mj}F_{jk}F_{km} + F_{km} \otimes F_{mj}F_{jk}F_{kk} + 1 \otimes F_{mj}F_{jk}F_{kk}F_{km} - mF_{km} \otimes F_{mj}F_{jk} - m1 \otimes F_{mj}F_{jk}F_{km} + F_{mj}F_{kk}F_{km} \otimes F_{jk} + F_{mj}F_{kk} \otimes F_{jk}F_{km} + F_{mj}F_{km} \otimes F_{jk}F_{kk} + F_{mj} \otimes F_{jk}F_{kk}F_{km} - mF_{mj}F_{km} \otimes F_{jk} - mF_{mj} \otimes F_{jk}F_{km} + F_{jk}F_{kk}F_{km} \otimes F_{mj} + F_{jk}F_{kk} \otimes F_{mj}F_{km} + F_{jk}F_{km} \otimes F_{mj}F_{kk} + F_{jk} \otimes F_{mj}F_{kk}F_{km} - mF_{jk}F_{km} \otimes F_{mj} - mF_{jk} \otimes F_{mj}F_{km}$
- $\Delta F_{mj}(F_{jj} - m)^2F_{jm}$
  - We have that $\Delta F_{mj}(F_{jj} - m) = F_{mj}F_{jj} \otimes 1 + F_{mj} \otimes F_{jj} - mF_{mj} \otimes 1 + F_{jj} \otimes F_{mj} + 1 \otimes F_{mj}F_{jj} - m1 \otimes F_{mj}$. We have that $\Delta(F_{jj} - m)F_{jm} = F_{jj}F_{jm} \otimes 1 + F_{jj} \otimes F_{jm} + F_{jm} \otimes F_{jj} + 1 \otimes F_{jj}F_{jm} - mF_{jm} \otimes 1 - m1 \otimes F_{jm}$.

Thus, we have the coproduct is: $F_{mj}F_{jj}^2F_{jm}\otimes 1+F_{mj}F_{jj}^2\otimes F_{jm}+F_{mj}F_{jj}F_{jm}\otimes F_{jj}+F_{mj}F_{jj}\otimes F_{jj}F_{jm}-mF_{mj}F_{jj}F_{jm}\otimes 1-mF_{mj}F_{jj}\otimes F_{jm}+F_{mj}F_{jj}F_{jm}\otimes F_{jj}+F_{mj}F_{jj}\otimes F_{jj}F_{jm}+F_{mj}F_{jm}\otimes F_{jj}^2+F_{mj}\otimes F_{jj}^2F_{jm}-mF_{mj}F_{jm}\otimes F_{jj}-mF_{mj}\otimes F_{jj}F_{jm}-mF_{mj}F_{jj}F_{jm}\otimes 1-mF_{mj}F_{jj}\otimes F_{jm}-mF_{mj}F_{jm}\otimes F_{jj}-mF_{mj}\otimes F_{jj}F_{jm}+m^2F_{mj}F_{jm}\otimes 1+m^2F_{mj}\otimes F_{jm}+F_{jj}^2F_{jm}\otimes F_{mj}+F_{jj}^2\otimes F_{mj}F_{jm}+F_{jj}F_{jm}\otimes F_{mj}F_{jj}+F_{jj}\otimes F_{mj}F_{jj}F_{jm}-mF_{jj}F_{jm}\otimes F_{mj}-mF_{jj}\otimes F_{mj}F_{jm}+F_{jj}F_{jm}\otimes F_{mj}F_{jj}+F_{jj}\otimes F_{mj}F_{jj}F_{jm}+F_{jm}\otimes F_{mj}F_{jj}^2+1\otimes F_{mj}F_{jj}^2F_{jm}-mF_{jm}\otimes F_{mj}F_{jj}-m1\otimes F_{mj}F_{jj}F_{jm}-mF_{jj}F_{jm}\otimes F_{mj}-mF_{jj}\otimes F_{mj}F_{jm}-mF_{jm}\otimes F_{mj}F_{jj}-m1\otimes F_{mj}F_{jj}F_{jm}+m^2F_{jm}\otimes F_{mj}+m^21\otimes F_{mj}F_{jm}$

From these calculations, we can obtain the following:

- $Q_t(F_{mm}-m)^4$
  - $F_{mm}^4+\langle F_{mm}^4\rangle_t+12F_{mm}^2t+4F_{mm}\langle F_{mm}^3\rangle_t-4mF_{mm}^3-4m\langle F_{mm}^3\rangle_t-24mF_{mm}t+6m^2F_{mm}^2+12m^2t-4m^3F_{mm}+m^4$
    - $\langle Q_t(F_{mm}-m)^4\rangle_\epsilon=\langle F_{mm}^4\rangle_\epsilon+\langle F_{mm}^4\rangle_t+12\langle F_{mm}^2\rangle_\epsilon t+4\langle F_{mm}\rangle_\epsilon\langle F_{mm}^3\rangle_t-4m\langle F_{mm}^3\rangle_\epsilon-4m\langle F_{mm}^3\rangle_t-24m\langle F_{mm}\rangle_\epsilon t+6m^2\langle F_{mm}^2\rangle_\epsilon+12m^2t-4m^3\langle F_{mm}\rangle_\epsilon+m^4$
- $Q_t(F_{mj}F_{jm}(F_{mm}-m)^2)$
  - $F_{mj}F_{jm}F_{mm}^2+\langle F_{mj}F_{jm}F_{mm}^2\rangle_t+2F_{mj}F_{jm}t+F_{mm}^2\langle F_{mj}F_{jm}\rangle_t+2F_{mm}\langle F_{mj}F_{jm}F_{mm}\rangle_t-2mF_{mj}F_{jm}F_{mm}-2m\langle F_{mj}F_{jm}F_{mm}\rangle_t-2mF_{mm}\langle F_{mj}F_{jm}\rangle_t+m^2F_{mj}F_{jm}+m^2\langle F_{mj}F_{jm}\rangle_t+F_{jm}\langle F_{mj}F_{mm}^2\rangle_t+F_{mj}\langle F_{jm}F_{mm}^2\rangle_t+2F_{mj}F_{mm}\langle F_{jm}F_{mm}\rangle_t+2F_{jm}F_{mm}\langle F_{mj}F_{mm}\rangle_t-2mF_{jm}\langle F_{mj}F_{mm}\rangle_t-2mF_{mj}\langle F_{jm}F_{mm}\rangle_t$
- $Q_t(F_{mm}-m)F_{mj}F_{jm}(F_{mm}-m)$
  - $F_{mm}F_{mj}F_{jm}F_{mm}+\langle F_{mm}F_{mj}F_{jm}F_{mm}\rangle_t+F_{mm}\langle F_{mm}F_{mj}F_{jm}\rangle_t-mF_{mm}F_{mj}F_{jm}-m\langle F_{mm}F_{mj}F_{jm}\rangle_t+F_{jm}\langle F_{mm}F_{mj}F_{mm}\rangle_t+F_{mm}F_{mj}\langle F_{jm}F_{mm}\rangle_t+F_{jm}F_{mm}\langle F_{mm}F_{mj}\rangle_t+mF_{jm}\langle F_{mm}F_{mj}\rangle_t+F_{mj}\langle F_{mm}F_{jm}F_{mm}\rangle_t+F_{mm}F_{jm}\langle F_{mj}F_{mm}\rangle_t+F_{mj}F_{mm}\langle F_{mm}F_{jm}\rangle_t-mF_{mj}\langle F_{mm}F_{jm}\rangle_t+F_{mm}^2\langle F_{mj}F_{jm}\rangle_t+2F_{mj}F_{jm}t+F_{mm}\langle F_{mj}F_{jm}F_{mm}\rangle_t+mF_{mm}\langle F_{mj}F_{jm}\rangle_t-mF_{mj}F_{jm}F_{mm}-m\langle F_{mj}F_{jm}F_{mm}\rangle_t-mF_{mm}\langle F_{mj}F_{jm}\rangle_t+m^2F_{mj}F_{jm}+m^2\langle F_{mj}F_{jm}\rangle_t-mF_{jm}\langle F_{mj}F_{mm}\rangle_t-mF_{mj}\langle F_{jm}F_{mm}\rangle_t$
- $Q_t(F_{mm}-m)^2F_{mj}F_{jm}$
  - $F_{mm}^2F_{mj}F_{jm}+\langle F_{mm}^2F_{mj}F_{jm}\rangle_t+2F_{mj}F_{jm}t+F_{mm}^2\langle F_{mj}F_{jm}\rangle_t+2F_{mm}\langle F_{mm}F_{mj}F_{jm}\rangle_t-2mF_{mm}F_{mj}F_{jm}-2m\langle F_{mm}F_{mj}F_{jm}\rangle_t-2mF_{mm}\langle F_{mj}F_{jm}\rangle_t+m^2F_{mj}F_{jm}+m^2\langle F_{mj}F_{jm}\rangle_t+F_{jm}\langle F_{mm}^2F_{mj}\rangle_t+F_{mj}\langle F_{mm}^2F_{jm}\rangle_t+2F_{mm}F_{mj}\langle F_{mm}F_{jm}\rangle_t+2F_{mm}F_{jm}\langle F_{mm}F_{mj}\rangle_t-2mF_{jm}\langle F_{mm}F_{mj}\rangle_t-2mF_{mj}\langle F_{mm}F_{jm}\rangle_t$
- $Q_t(F_{mm}-m)F_{mj}F_{jk}F_{km}$
  - $F_{mm}F_{mj}F_{jk}F_{km}+F_{mm}F_{mj}\langle F_{jk}F_{km}\rangle_t+F_{mm}F_{jk}\langle F_{mj}F_{km}\rangle_t+F_{mm}F_{km}\langle F_{mj}F_{jk}\rangle_t+F_{mm}\langle F_{mj}F_{jk}F_{km}\rangle_t+F_{mj}F_{jk}\langle F_{mm}F_{km}\rangle_t+F_{mj}F_{km}\langle F_{mm}F_{jk}\rangle_t+F_{mj}\langle F_{mm}F_{jk}F_{km}\rangle_t+F_{jk}F_{km}\langle F_{mm}F_{mj}\rangle_t+F_{jk}\langle F_{mm}F_{mj}F_{km}\rangle_t+F_{km}\langle F_{mm}F_{mj}F_{jk}\rangle_t+\langle F_{mm}F_{mj}F_{jk}F_{km}\rangle_t-mF_{mj}F_{jk}F_{km}-mF_{mj}\langle F_{jk}F_{km}\rangle_t-mF_{jk}\langle F_{mj}F_{jk}\rangle_t-mF_{km}\langle F_{mj}F_{jk}\rangle_t-m\langle F_{mj}F_{jk}F_{km}\rangle_t$
- $Q_t(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$
  - $F_{mm}F_{mj}F_{jj}F_{jm}+F_{mm}F_{mj}\langle F_{jj}F_{jm}\rangle_t-mF_{mm}F_{mj}F_{jm}+F_{mm}F_{jj}\langle F_{mj}F_{jm}\rangle_t+F_{mm}F_{jm}\langle F_{mj}F_{jj}\rangle_t+F_{mm}\langle F_{mj}F_{jj}F_{jm}\rangle_t-mF_{mm}\langle F_{mj}F_{jm}\rangle_t+F_{mj}F_{jj}\langle F_{mm}F_{jm}\rangle_t+F_{mj}F_{jm}\langle F_{mm}F_{jj}\rangle_t+F_{mj}\langle F_{mm}F_{jj}F_{jm}\rangle_t-mF_{mj}\langle F_{mm}F_{jm}\rangle_t+F_{jj}F_{jm}\langle F_{mm}F_{mj}\rangle_t+F_{jj}\langle F_{mm}F_{mj}F_{jm}\rangle_t+F_{jm}\langle F_{mm}F_{mj}F_{jj}\rangle_t+\langle F_{mm}F_{mj}F_{jj}F_{jm}\rangle_t-mF_{jm}\langle F_{mm}F_{mj}\rangle_t-m\langle F_{mm}F_{mj}F_{jm}\rangle_t-mF_{mj}F_{jj}F_{jm}-mF_{mj}\langle F_{jj}F_{jm}\rangle_t+m^2F_{mj}F_{jm}-mF_{jj}\langle F_{mj}F_{jm}\rangle_t-mF_{jm}\langle F_{mj}F_{jj}\rangle_t-m\langle F_{mj}F_{jj}F_{jm}\rangle_t+m^2\langle F_{mj}F_{jm}\rangle_t$

- $Q_tF_{mj}F_{jm}F_{mk}F_{km}$
  - $F_{mj}F_{jm}F_{mk}F_{km}+F_{mj}F_{jm}\langle F_{mk}F_{km}\rangle_t+F_{mk}F_{km}\langle F_{mj}F_{jm}\rangle_t+\langle F_{mj}F_{jm}F_{mk}F_{km}\rangle_t+F_{mk}\langle F_{mj}F_{jm}F_{km}\rangle_t+F_{km}\langle F_{mj}F_{jm}F_{mk}\rangle_t+F_{mj}\langle F_{jm}F_{mk}F_{km}\rangle_t+F_{mj}F_{mk}\langle F_{jm}F_{km}\rangle_t+F_{mj}F_{km}\langle F_{jm}F_{mk}\rangle_t+F_{jm}\langle F_{mj}F_{mk}F_{km}\rangle_t+F_{jm}F_{mk}\langle F_{mj}F_{km}\rangle_t+F_{jm}F_{km}\langle F_{mj}F_{mk}\rangle_t$
- $Q_tF_{mj}F_{jk}F_{km}(F_{mm}-m)$
  - $F_{mj}F_{jk}F_{km}F_{mm}-mF_{mj}F_{jk}F_{km}+F_{mj}F_{jk}\langle F_{km}F_{mm}\rangle_t+F_{km}F_{mm}\langle F_{mj}F_{jk}\rangle_t+F_{km}\langle F_{mj}F_{jk}F_{mm}\rangle_t-mF_{km}\langle F_{mj}F_{jk}\rangle_t+F_{mm}\langle F_{mj}F_{jk}F_{km}\rangle_t+\langle F_{mj}F_{jk}F_{km}F_{mm}\rangle_t-m\langle F_{mj}F_{jk}F_{km}\rangle_t+F_{mj}F_{km}\langle F_{jk}F_{mm}\rangle_t+F_{mj}F_{mm}\langle F_{jk}F_{km}\rangle_t+F_{mj}\langle F_{jk}F_{km}F_{mm}\rangle_t-mF_{mj}\langle F_{jk}F_{km}\rangle_t+F_{jk}F_{km}\langle F_{mj}F_{mm}\rangle_t+F_{jk}F_{mm}\langle F_{mj}F_{km}\rangle_t+F_{jk}\langle F_{mj}F_{km}F_{mm}\rangle_t-mF_{jk}\langle F_{mj}F_{km}\rangle_t$
- $Q_tF_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$
  - $F_{mj}F_{jj}F_{jm}F_{mm}-mF_{mj}F_{jj}F_{jm}+F_{mj}F_{jj}\langle F_{jm}F_{mm}\rangle_t+F_{mj}F_{jm}\langle F_{jj}F_{mm}\rangle_t+F_{mj}F_{mm}\langle F_{jj}F_{jm}\rangle_t+F_{mj}\langle F_{jj}F_{jm}F_{mm}\rangle_t-mF_{mj}\langle F_{jj}F_{jm}\rangle_t-mF_{mj}F_{jm}F_{mm}+m^2F_{mj}F_{jm}-mF_{mj}\langle F_{jm}F_{mm}\rangle_t+F_{jj}F_{jm}\langle F_{mj}F_{mm}\rangle_t+F_{jj}F_{mm}\langle F_{mj}F_{jm}\rangle_t+F_{jj}\langle F_{mj}F_{jm}F_{mm}\rangle_t-mF_{jj}\langle F_{mj}F_{jm}\rangle_t+F_{jm}F_{mm}\langle F_{mj}F_{jj}\rangle_t+F_{jm}\langle F_{mj}F_{jj}F_{mm}\rangle_t-mF_{jm}\langle F_{mj}F_{jj}\rangle_t+F_{mm}\langle F_{mj}F_{jj}F_{jm}\rangle_t+\langle F_{mj}F_{jj}F_{jm}F_{mm}\rangle_t-m\langle F_{mj}F_{jj}F_{jm}\rangle_t-mF_{jm}\langle F_{mj}F_{mm}\rangle_t-mF_{mm}\langle F_{mj}F_{jm}\rangle_t-m\langle F_{mj}F_{jm}F_{mm}\rangle_t+m^2\langle F_{mj}F_{jm}\rangle_t$
- $Q_tF_{mj}F_{jk}F_{kl}F_{lm}$
  - $F_{mj}F_{jk}F_{kl}F_{lm}+F_{mj}F_{jk}\langle F_{kl}F_{lm}\rangle_t+F_{kl}F_{lm}\langle F_{mj}F_{jk}\rangle_t+\langle F_{mj}F_{jk}F_{kl}F_{lm}\rangle_t+F_{kl}\langle F_{mj}F_{jk}F_{lm}\rangle_t+F_{lm}\langle F_{mj}F_{jk}F_{kl}\rangle_t+F_{mj}\langle F_{jk}F_{kl}F_{lm}\rangle_t+F_{mj}F_{kl}\langle F_{jk}F_{lm}\rangle_t+F_{mj}F_{lm}\langle F_{jk}F_{kl}\rangle_t+F_{jk}\langle F_{mj}F_{kl}F_{lm}\rangle_t+F_{jk}F_{kl}\langle F_{mj}F_{lm}\rangle_t+F_{jk}F_{lm}\langle F_{mj}F_{kl}\rangle_t$
- $Q_tF_{mj}(F_{jj}-m)F_{jl}F_{lm}$
  - $F_{mj}F_{jj}F_{jl}F_{lm}+F_{mj}F_{jj}\langle F_{jl}F_{lm}\rangle_t+F_{mj}\langle F_{jj}F_{jl}F_{lm}\rangle_t+F_{mj}F_{jl}\langle F_{jj}F_{lm}\rangle_t+F_{mj}F_{lm}\langle F_{jj}F_{jl}\rangle_t-mF_{mj}F_{jl}F_{lm}-mF_{mj}\langle F_{jl}F_{lm}\rangle_t+F_{jj}\langle F_{mj}F_{jl}F_{lm}\rangle_t+F_{jj}F_{jl}\langle F_{mj}F_{lm}\rangle_t+F_{jj}F_{lm}\langle F_{mj}F_{jl}\rangle_t+F_{jl}F_{lm}\langle F_{mj}F_{jj}\rangle_t+\langle F_{mj}F_{jj}F_{jl}F_{lm}\rangle_t+F_{jl}\langle F_{mj}F_{jj}F_{lm}\rangle_t+F_{lm}\langle F_{mj}F_{jj}F_{jl}\rangle_t-m\langle F_{mj}F_{jl}F_{lm}\rangle_t-mF_{jl}\langle F_{mj}F_{lm}\rangle_t-mF_{lm}\langle F_{mj}F_{jl}\rangle_t$
- $Q_tF_{mj}F_{jk}F_{kj}F_{jm}$
  - $F_{mj}F_{jk}F_{kj}F_{jm}+F_{mj}F_{jk}\langle F_{kj}F_{jm}\rangle_t+F_{kj}F_{jm}\langle F_{mj}F_{jk}\rangle_t+\langle F_{mj}F_{jk}F_{kj}F_{jm}\rangle_t+F_{kj}\langle F_{mj}F_{jk}F_{jm}\rangle_t+F_{jm}\langle F_{mj}F_{jk}F_{kj}\rangle_t+F_{mj}\langle F_{jk}F_{kj}F_{jm}\rangle_t+F_{mj}F_{kj}\langle F_{jk}F_{jm}\rangle_t+F_{mj}F_{jm}\langle F_{jk}F_{kj}\rangle_t+F_{jk}\langle F_{mj}F_{kj}F_{jm}\rangle_t+F_{jk}F_{kj}\langle F_{mj}F_{jm}\rangle_t+F_{jk}F_{jm}\langle F_{mj}F_{kj}\rangle_t$
- $Q_tF_{mj}F_{jk}(F_{kk}-m)F_{km}$
  - $F_{mj}F_{jk}F_{kk}F_{km}+F_{mj}F_{jk}\langle F_{kk}F_{km}\rangle_t-mF_{mj}F_{jk}F_{km}+F_{kk}F_{km}\langle F_{mj}F_{jk}\rangle_t+F_{kk}\langle F_{mj}F_{jk}F_{km}\rangle_t+F_{km}\langle F_{mj}F_{jk}F_{kk}\rangle_t+\langle F_{mj}F_{jk}F_{kk}F_{km}\rangle_t-mF_{km}\langle F_{mj}F_{jk}\rangle_t-m\langle F_{mj}F_{jk}F_{km}\rangle_t+F_{mj}F_{kk}\langle F_{jk}F_{km}\rangle_t+F_{mj}F_{km}\langle F_{jk}F_{kk}\rangle_t+F_{mj}\langle F_{jk}F_{kk}F_{km}\rangle_t-mF_{mj}\langle F_{jk}F_{km}\rangle_t+F_{jk}F_{kk}\langle F_{mj}F_{km}\rangle_t+F_{jk}F_{km}\langle F_{mj}F_{kk}\rangle_t+F_{jk}\langle F_{mj}F_{kk}F_{km}\rangle_t-mF_{jk}\langle F_{mj}F_{km}\rangle_t$
- $Q_tF_{mj}(F_{jj}-m)^2F_{jm}$
  - $F_{mj}F_{jj}^2F_{jm}+F_{mj}F_{jj}\langle F_{jj}F_{jm}\rangle_t-mF_{mj}F_{jj}F_{jm}+F_{mj}F_{jj}\langle F_{jj}F_{jm}\rangle_t+F_{mj}F_{jm}\langle F_{jj}^2\rangle_t+F_{mj}\langle F_{jj}^2F_{jm}\rangle_t-mF_{mj}\langle F_{jj}F_{jm}\rangle_t-mF_{mj}F_{jj}F_{jm}-mF_{mj}\langle F_{jj}F_{jm}\rangle_t+m^2F_{mj}F_{jm}+F_{jj}^2\langle F_{mj}F_{jm}\rangle_t+F_{jj}F_{jm}\langle F_{mj}F_{jj}\rangle_t+F_{jj}\langle F_{mj}F_{jj}F_{jm}\rangle_t-mF_{jj}\langle F_{mj}F_{jm}\rangle_t+F_{jj}F_{jm}\langle F_{mj}F_{jj}\rangle_t+F_{jj}\langle F_{mj}F_{jj}F_{jm}\rangle_t+F_{jm}\langle F_{mj}F_{jj}^2\rangle_t+\langle F_{mj}F_{jj}^2F_{jm}\rangle_t-mF_{jm}\langle F_{mj}F_{jj}\rangle_t-m\langle F_{mj}F_{jj}F_{jm}\rangle_t-mF_{jj}\langle F_{mj}F_{jm}\rangle_t-mF_{jm}\langle F_{mj}F_{jj}\rangle_t-m\langle F_{mj}F_{jj}F_{jm}\rangle_t+m^2\langle F_{mj}F_{jm}\rangle_t$

We have that $Q_t\Phi_4=\sum_{m=1}^{n}Q_t\Phi_4^{(m)}+Q_t\widetilde{\Phi_4^{(m)}}$. Note that $Q_t\Phi_4^{(m)}$ is the sum of:

- $Q_t(F_{mm}-m)^4$
- $\sum_{j=-m}^{m-1}\frac{4}{3}Q_t(F_{mj}F_{jm}(F_{mm}-m)^2))$

- $\sum_{j=-m}^{m-1} \frac{4}{3}Q_t(F_{mm}-m)F_{mj}F_{jm}(F_{mm}-m)$
- $\sum_{j=-m}^{m-1} \frac{4}{3}(F_{mm}-m)^2F_{mj}F_{jm}$
- where $j \neq k$, $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2Q_t(F_{mm}-m)F_{mj}F_{jk}F_{km}$
- $\sum_{j=-m}^{m-1} 2Q_t(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$
- $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2Q_tF_{mj}F_{jm}F_{mk}F_{km}$
- where $j \neq k$, $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 2Q_tF_{mj}F_{jk}F_{km}(F_{mm}-m)$
- $\sum_{j=-m}^{m-1} 2Q_tF_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$
- where $j \neq k \neq l$, $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\sum_{l=-m}^{m-1} 4Q_tF_{mj}F_{jk}F_{kl}F_{lm}$
- where $j \neq l$, $\sum_{j=-m}^{m-1}\sum_{l=-m}^{m-1} 4Q_tF_{mj}(F_{jj}-m)F_{jl}F_{lm}$
- where $j \neq k$, $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 4Q_tF_{mj}F_{jk}F_{kj}F_{jm}$
- where $j \neq k$, $\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1} 4Q_tF_{mj}F_{jk}(F_{kk}-m)F_{km}$
- $\sum_{j=-m}^{m-1} 4Q_tF_{mj}(F_{jj}-m)^2F_{jm}$

and $Q_t\widetilde{\Phi_4^{(m)}}$ is the sum of:

- $Q_t(F_{mm}-m)^4$
- $\sum_{j=-m+1}^{m-1} \frac{4}{3}Q_t(F_{mj}F_{jm}(F_{mm}-m)^2))$
- $\sum_{j=-m+1}^{m-1} \frac{4}{3}Q_t(F_{mm}-m)F_{mj}F_{jm}(F_{mm}-m)$
- $\sum_{j=-m+1}^{m-1} \frac{4}{3}(F_{mm}-m)^2F_{mj}F_{jm}$
- where $j \neq k$, $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} -2Q_t(F_{mm}-m)F_{mj}F_{jk}F_{km}$
- $\sum_{j=-m+1}^{m-1} 2Q_t(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$
- $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 2Q_tF_{mj}F_{jm}F_{mk}F_{km}$
- where $j \neq k$, $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} -2Q_tF_{mj}F_{jk}F_{km}(F_{mm}-m)$
- $\sum_{j=-m+1}^{m-1} 2Q_tF_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$
- where $j \neq k \neq l$, $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\sum_{l=-m+1}^{m-1} 4Q_tF_{mj}F_{jk}F_{kl}F_{lm}$
- where $j \neq l$, $\sum_{j=-m+1}^{m-1}\sum_{l=-m+1}^{m-1} -4Q_tF_{mj}(F_{jj}-m)F_{jl}F_{lm}$
- where $j \neq k$, $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} 4Q_tF_{mj}F_{jk}F_{kj}F_{jm}$
- where $j \neq k$, $\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1} -4Q_tF_{mj}F_{jk}(F_{kk}-m)F_{km}$
- $\sum_{j=-m+1}^{m-1} 4Q_tF_{mj}(F_{jj}-m)^2F_{jm}$

Thus, $\lim_{\epsilon\to 0}\langle\Phi_4\rangle_{t+\epsilon} = \lim_{\epsilon\to 0}\langle Q_t\Phi_4\rangle_\epsilon = \lim_{\epsilon\to 0}\sum_{m=1}^{n}\langle Q_t\Phi_4^{(m)}\rangle_\epsilon + \langle Q_t\widetilde{\Phi_4^{(m)}}\rangle_\epsilon$, which will be the sum of, for $m = 1$ to n... $\lim_{\epsilon\to 0}\langle Q_t\Phi_4^{(m)}\rangle_\epsilon$:

- $\langle F_{mm}^4\rangle_t - 4m\langle F_{mm}^3\rangle_t + 12m^2t + m^4$
  $-\ (2t+12t^2) + 0 + 12m^2t + m^4$
- $\frac{4}{3}\sum_{j=-m}^{m-1}\langle F_{mj}F_{jm}F_{mm}^2\rangle_t - 2m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}[(4t+8t^2)-2m(4t)+m^2(4t)] + \frac{4}{3}\sum_{j=-m+1}^{m+1}(t+4t^2)-2m(t)+m^2(2t)$
- $\frac{4}{3}\sum_{j=-m}^{m-1}\langle F_{mm}F_{mj}F_{jm}F_{mm}\rangle_t - m\langle F_{mm}F_{mj}F_{jm}\rangle_t - m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}(4t+8t^2-4mt-4mt+m^2(4t)) + \frac{4}{3}\sum_{j=-m+1}^{m-1} t+4t^2-mt-mt+m^2(2t)$
- $\frac{4}{3}\sum_{j=-m}^{m-1}\langle F_{mm}^2F_{mj}F_{jm}\rangle_t - 2m\langle F_{mm}F_{mj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}(4t+8t^2-2m(4t)+m^2(4t)) + \frac{4}{3}\sum_{j=-m+1}^{m-1} t+4t^2-2mt+m^2(2t)$

- where $j \neq k$, $2\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\langle F_{mm}F_{mj}F_{jk}F_{km}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  $-\ 6(m-1)(2)(2t-2mt)+(4m-8)(m-1)(2)(t-mt)$
- $2\sum_{j=-m}^{m-1}\langle F_{mm}F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mm}F_{mj}F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ 2(4t-8t^2-m(4t)-m(4t)+m^2(4t))+2\sum_{j=-m+1}^{m-1} t-mt-mt+m^2(2t)$
- $2\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\langle F_{mj}F_{jm}F_{mk}F_{km}\rangle_t$
  $-\ 2(16t+32t^2)+4(m-1)2(4t+8t^2)+2(m-1)2(2t+8t^2)+(2m-2)(2m-3)2(t+4t^2)$
- where $j \neq k$, $2\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\langle F_{mj}F_{jk}F_{km}F_{mm}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  $-\ 6(m-1)2(2t-2mt)+(4m-8)(m-1)2(t-mt)$
- $2\sum_{j=-m}^{m-1}\langle F_{mj}F_{jj}F_{jm}F_{mm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ 2(4t-8t^2)-2m(4t)-2m(4t)+2m^2(4t)+2\sum_{j=-m+1}^{m-1} t-mt-mt+m^2(2t)$
- where $j \neq k \neq l$, $4\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\sum_{l=-m}^{m-1}\langle F_{mj}F_{jk}F_{kl}F_{lm}\rangle_t$
  $-\ 2(m-1)4(t-4t^2)+2(m-1)4(4t)+((2m-2)(2m-4))4(2t)+(2m-2)(2m-4)4(2t)+(2m-2)4(4t)+((2m-2)(2m-4))4(2t)+(2m-2)(2m-4)4(2t)+(m-1)(8m^2-32m+32)4t$
- where $j \neq l$, $4\sum_{j=-m}^{m-1}\sum_{l=-m}^{m-1}\langle F_{mj}F_{jj}F_{jl}F_{lm}\rangle_t - m\langle F_{mj}F_{jl}F_{lm}\rangle_t$
  $-\ 6(m-1)4(2t-2mt)+(4m-8)(m-1)4(t-mt)$
- where $j \neq k$, $4\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\langle F_{mj}F_{jk}F_{kj}F_{jm}\rangle_t$
  $-\ 2(m-1)4(4t+8t^2)+2(m-1)4(4t+8t^2)+(4m-6)(m-1)4(t+4t^2)$
- where $j \neq k$, $4\sum_{j=-m}^{m-1}\sum_{k=-m}^{m-1}\langle F_{mj}F_{jk}F_{kk}F_{km}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  $-\ 6(m-1)4(2t-2mt)+(4m-8)(m-1)4(t-mt)$
- $4\sum_{j=-m}^{m-1}\langle F_{mj}F_{jj}^2F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ 4(4t+8t^2-2m(4t)+m^2(4t))+4\sum_{j=-m+1}^{m-1} t+4t^2-2mt+m^2(2t)$

and the sum of $\lim_{\epsilon\to 0}\langle Q_t\widetilde{\Phi_4^{(m)}}\rangle_\epsilon$:

- $\langle F_{mm}^4\rangle_t - 4m\langle F_{mm}^3\rangle_t + 12m^2t + m^4$
  $-\ 2t+12t^2+12m^2t+m^4$
- $\frac{4}{3}\sum_{j=-m+1}^{m-1}\langle F_{mj}F_{jm}F_{mm}^2\rangle_t - 2m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}\sum_{j=-m+1}^{m-1}(t+4t^2)-2m(t)+m^2(2t)$
- $\frac{4}{3}\sum_{j=-m+1}^{m-1}\langle F_{mm}F_{mj}F_{jm}F_{mm}\rangle_t - m\langle F_{mm}F_{mj}F_{jm}\rangle_t - m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}\sum_{j=-m+1}^{m-1} t+4t^2-2mt+m^2(2t)$
- $\frac{4}{3}\sum_{j=-m+1}^{m-1}\langle F_{mm}^2F_{mj}F_{jm}\rangle_t - 2m\langle F_{mm}F_{mj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ \frac{4}{3}\sum_{j=-m+1}^{m-1} t+4t^2-2mt+m^2(2t)$
- where $j \neq k$, $-2\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\langle F_{mm}F_{mj}F_{jk}F_{km}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  $-\ 2(m-1)(-2)(2t-2mt)+(4m-8)(m-1)(-2)(t-mt)$
- $2\sum_{j=-m+1}^{m-1}\langle F_{mm}F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mm}F_{mj}F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  $-\ 2\sum_{j=-m+1}^{m-1} t-2mt+m^2(2t)$
- $2\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\langle F_{mj}F_{jm}F_{mk}F_{km}\rangle_t$
  $-\ 2(m-1)2(2t+8t^2)+(2m-2)(2m-3)2(t+4t^2)$
- where $j \neq k$, $-2\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\langle F_{mj}F_{jk}F_{km}F_{mm}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  $-\ 2(m-1)(-2)(2t-2mt)+(4m-8)(m-1)(-2)(t-mt)$
- $2\sum_{j=-m+1}^{m-1}\langle F_{mj}F_{jj}F_{jm}F_{mm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mj}F_{jm}F_{mm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$

  - $2\sum_{j=-m+1}^{m-1} t - 2mt + m^2(2t)$
- where $j \neq k \neq l$, $4\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\sum_{l=-m+1}^{m-1}\langle F_{mj}F_{jk}F_{kl}F_{lm}\rangle_t$
  - $2(2m-2)(2m-4)(4)(2t) + (2m-2)(2m-4)(2m-5)(4)(t)$
- where $j \neq l$, $-4\sum_{j=-m+1}^{m-1}\sum_{l=-m+1}^{m-1}\langle F_{mj}F_{jj}F_{jl}F_{lm}\rangle_t - m\langle F_{mj}F_{jl}F_{lm}\rangle_t$
  - $(2m-2)(-4)(2t-2mt) + (2m-2)(2m-4)(-4)(t-mt)$
- where $j \neq k$, $4\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\langle F_{mj}F_{jk}F_{kj}F_{jm}\rangle_t$
  - $(2m-2)(4)(4t+8t^2) + (2m-2)(2m-4)(4)(t+4t^2)$
- where $j \neq k$, $-4\sum_{j=-m+1}^{m-1}\sum_{k=-m+1}^{m-1}\langle F_{mj}F_{jk}F_{kk}F_{km}\rangle_t - m\langle F_{mj}F_{jk}F_{km}\rangle_t$
  - $(2m-2)(-4)(2t-2mt) + (2m-2)(2m-4)(-4)(t-mt)$
- $4\sum_{j=-m+1}^{m-1}\langle F_{mj}F_{jj}^2F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t - m\langle F_{mj}F_{jj}F_{jm}\rangle_t + m^2\langle F_{mj}F_{jm}\rangle_t$
  - $4\sum_{j=-m+1}^{m-1} t + 4t^2 - 2mt + m^2(2t)$

So we have that the expression within the $\sum_{m=1}^{n}$ is the sum of:

- constants
  - $2m^4$
- t
  - $4+24m^2+32-64m+32m^2+48(1-2m+2m^2)(m-1)+48(m-1)(2-2m)+8-16m+8m^2+32m+(m-1)(16m-8)+8-16m+8m^2+72(m-1)+32(2m-2)(2m-4)+4(m-1)(8m^2-32m+32)+(2m-2)(2m-4)(8m-4)+96m-96+4(4m-6)(m-1)+4(2m-2)(2m-4)$
- t^2
  - $24+32+32(2m-2)-16+64m+(m-1)(64m-32)-16-32(m-1)+192(m-1)+16(4m-6)(m-1)+16(2m-2)(2m-4)+32+64(m-1)$

.

This simplifies to

- constants
  - $2m^4$
- t
  - $160m^3 - 312m^2 + 312m - 100$
- t^2
  - $192m^2 - 96m + 24$

So finally we have that $\langle\Phi_4\rangle_t = \sum_{m=1}^{n} 2m^4 + t\sum_{m=1}^{n}(160m^3 - 312m^2 + 312m - 100) + t^2(\sum_{m=1}^{n} 192m^2 - 96m + 24) = \frac{n(n+1)(6n^3+9n^2+n-1)}{15} + t(160\frac{n^2(n+1)^2}{4} - 312\frac{n(n+1)(2n+1)}{6} + 312\frac{n(n+1)}{2} - 100n) + t^2(192\frac{n(n+1)(2n+1)}{6} - 96\frac{n(n+1)}{2} + 24n)$.

This simplifies to $\langle\Phi_4\rangle_t = (64n^3 + 48n^2 + 8n)t^2 + (40n^4 - 24n^3 + 40n^2 + 4n)t + \frac{6n^5+15n^4+10n^3-n}{15}$

Remark 4.5. From Proposition 4.4 of [1], we have that $Q_t\Phi_4 \in Z(U(sp_{2n}))$. It is clear that we must have $Q_t\Phi_4 = \Phi_4 + a\Phi_2 + b\Phi_2^2 + c$, for some constants a, b, c in terms of n and t. We claim that $b = 0$. To see this, note that among Φ_4, Φ_2, Φ_2^2, we have that Φ_2^2 is the only one that contains terms of the form $F_{mm}F_{jj}$, where $j \neq k \in Z^+$. However, if we consider Remark 4.2, the only terms that could yield $F_{mm}F_{jj}$ would be from terms such as $2(F_{mm}-m)F_{mj}(F_{jj}-m)F_{jm}$ and $2F_{mj}(F_{jj}-m)F_{jm}(F_{mm}-m)$. However, if we apply Q_t to these terms and produce a $F_{mm}F_{jj}$, there will be a symmetrical $F_{mm}F_{-j,-j} = -F_{mm}F_{jj}$. Thus, these terms will cancel, so we must have $Q_t\Phi_4 = \Phi_4 = \Phi_4 + a\Phi_2 + c$. To find what a is, it suffices to find the coefficient of F_{mm}^2 in $Q_t\Phi_4$ for a particular m, and subtract the

F_{mm}^2 terms contained in Φ_4, then divide by 2, since the coefficient of F_{mm}^2 in Φ_2 is 2.

This yields $Q_t\Phi_4 = \Phi_4 + (16nt+4t)\Phi_2 + C$ for some constant C. We have that $\langle Q_t\Phi_4\rangle_0 = \langle\Phi_4\rangle_t = \langle\Phi_4\rangle_0 + (16nt+4t)\langle\Phi_2\rangle_0 + C$. So $C = (64n^3+48n^2+8n)t^2 + (40n^4-24n^3+40n^2+4n)t + \frac{6n^5+15n^4+10n^3-n}{15} - \frac{6n^5+15n^4+10n^3-n}{15} - (16nt+4t)(\frac{n(n+1)(2n+1)}{3})$.

So $Q_t\Phi_4 = \Phi_4 + (16nt+4t)\Phi_2 + (64n^3+48n^2+8n)t^2 + (40n^4-24n^3+40n^2+4n)t - (16nt+4t)(\frac{n(n+1)(2n+1)}{3})$

Remark 4.6. We have

$$\Phi_2^{(\eta_1 L)} = \sum_{m=1}^{\eta_1 L} 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2F_{m,-m}F_{-m,m} + \sum_{j=-m+1}^{m-1} 4F_{mj}F_{jm}$$

,

$$\langle\Phi_2^{(\eta_1 L)}\rangle_{\tau_1 L} = \frac{\eta_1 L(\eta_1 L+1)(2\eta_1 L+1)}{3} + (8\eta_1^2L^2 + 4\eta_1 L)\tau_1 L$$

and $Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} = \sum_{m=1}^{\eta_2 L} 2\langle F_{mm}^2\rangle_{(\tau_2-\tau_1)L} + 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2\langle F_{m,-m}F_{-m,m}\rangle_{(\tau_2-\tau_1)L} + 2F_{m,-m}F_{-m,m} + 4\sum_{j=-m+1}^{m-1}\langle F_{mj}F_{jm}\rangle_{(\tau_2-\tau_1)L} + F_{mj}F_{jm}$

so

$$Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)} =$$

$$\sum_{m=1}^{\eta_2 L} 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2F_{m,-m}F_{-m,m} + \sum_{j=-m+1}^{m-1} 4F_{mj}F_{jm} + 8(\tau_2-\tau_1)\eta_2^2L^3 + 4(\tau_2-\tau_1)\eta_2 L^2$$

and we also have $\langle Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)}\rangle_{\tau_1 L} = \langle\Phi_2^{(\eta_2 L)}\rangle_{\tau_2 L} =$

$$\frac{\eta_2 L(\eta_2 L+1)(2\eta_2 L+1)}{3} + (8\eta_2^2L^2 + 4\eta_2)\tau_2 L$$

We want to calculate $\lim_{L\to\infty} L^{-2}$ of

$$\langle\Phi_2^{(\eta_1 L)}Q_{(\tau_2-\tau_1)L}\Phi_2^{(\eta_2 L)}\rangle_{\tau_1 L}$$
$$-\langle\Phi_2^{(\eta_1 L)}\rangle_{\tau_1 L}\langle\Phi_2^{(\eta_2 L)}\rangle_{\tau_2 L} =$$

$[8(\tau_2-\tau_1)\eta_2^2L^3 + 4(\tau_2-\tau_1)\eta_2L^2]\langle\Phi_2^{(\eta_1 L)}\rangle_{\tau_1 L} + \langle(\sum_{m=1}^{\eta_1 L} 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2F_{m,-m}F_{-m,m} + \sum_{j=-m+1}^{m-1} 4F_{mj}F_{jm}) * (\sum_{m=1}^{\eta_2 L} 2F_{mm}^2 - 4mF_{mm} + 2m^2 + 2F_{m,-m}F_{-m,m} + \sum_{j=-m+1}^{m-1} 4F_{mj}F_{jm})\rangle_{\tau_1 L} - \langle\Phi_2^{(\eta_1 L)}\rangle_{\tau_1 L}\langle\Phi_2^{(\eta_2 L)}\rangle_{\tau_2 L}$.

A long multiplication and keeping track of various states yields:

$$[8(\tau_2-\tau_1)\eta_2^2L^3 + 4(\tau_2-\tau_1)\eta_2L^2] * [\frac{\eta_1 L(\eta_1 L+1)(2\eta_1 L+1)}{3} + (8\eta_1^2L^2 + 4\eta_1 L)\tau_1 L]$$

$$+4\eta L(2\tau_1 L + 12\tau_1^2L^2) + 4(4\tau_1^2L^2)(\eta_1\eta_2L^2 - \eta L) + 4\eta_1\tau_1 L^2\frac{\eta_2 L(\eta_2 L+1)(2\eta_2 L+1)}{3}$$

$$+4\eta L(4\tau_1 L + 8\tau_1^2L^2) + 4(8\tau_1^2L^2)(\eta_1\eta_2L^2 - \eta L)$$

$$+(2\eta\eta_2L^2-2\eta L)8(\tau_1L+4\tau_1^2L^2)+(\eta_1L(\eta_2L-1)\eta_2L-2\eta_1\eta_2L^2+2\eta_1L)8(4\tau_1^2L^2)$$

$$+(\frac{16\eta L(\eta L+1)(2\eta L+1)}{3})\tau_1L$$

$$-16\eta L(\eta L+1)\tau_1L$$

$$-(\frac{16\eta L(\eta L+1)(2\eta L+1)}{3}-16\eta L(\eta L+1))(\tau_1L)$$

$$+(\frac{\eta_1L(\eta_1L+1)(2\eta_1L+1)}{3})(\eta_2L(4\tau_1L)+\frac{\eta_2L(\eta_2L+1)(2\eta_2L+1)}{3}+\eta_2L(8\tau_1L)+8\tau_1L(\eta_2L-1)\eta_2L)$$

$$+4\eta L(4\tau_1L+8\tau_1^2L^2)+(\eta_1\eta_2L^2-\eta L)4(8\tau_1^2L^2)$$

$$-16(\eta L)(\eta L+1)\tau_1L$$

$$+(2\eta_1L)(4\tau_1L)\frac{\eta_2L(\eta_2L+1)(2\eta_2L+1)}{3}$$

$$+4\eta L(16\tau_1L+32\tau_1^2L^2)+4(\eta_1\eta_2L^2-\eta L)(16\tau_1^2L^2)$$

$+8(4\tau_1L+8\tau_1^2L^2)(\eta L-1)\eta L+4(4\tau_1L+8\tau_1^2L^2)(2\eta\eta_2L^2-\eta L(\eta L+1))+8(\eta_1(\eta_2L-1)\eta_2L-(\eta L-1)\eta L-\eta\eta_2L^2+\frac{\eta L(\eta L+1)}{2})(8\tau_1^2L^2)$

$$+(2\eta_1\eta L^2-2\eta L)8(\tau_1L+4\tau_1^2L^2)+((\eta_2L)(\eta_1L-1)(\eta_1L)-2\eta_1\eta L^2+2\eta L)8(4\tau_1^2L^2)$$

$$-(\frac{16\eta L(\eta L+1)(2\eta L+1)}{3}-16\eta L(\eta L+1))(\tau_1L)$$

$$+\frac{\eta_2L(\eta_2L+1)(2\eta_2L+1)}{3}(\eta_1L-1)\eta_1L(8\tau_1L)$$

$+((\eta L-1)\eta L+\eta_1\eta L^2-\frac{\eta L(\eta L+1)}{2})8(4\tau_1L+8\tau_1^2L^2)+(\eta_1\eta_2L^2(\eta_1L-1)-(\eta L-1)\eta L-\eta_1\eta L^2+\frac{\eta L(\eta L+1)}{2})8(8\tau_1^2L^2)$

$$+\eta L(\eta L-1)16(2\tau_1L+8\tau_1^2L^2)+(\frac{2\eta L(\eta L+1)(2\eta L+1)}{3}-5\eta L(\eta L+1)+6\eta L)16(\tau_1L)+$$

$+(-2\eta L(\eta L-1)+\frac{(\eta L-1)\eta L(2\eta L-1)}{3}+(\eta L-1)\eta L+2\eta\eta_2L^2(\eta L-1)-\eta_2L(\eta L-1)\eta L+2\eta_1\eta L^2(\eta L-1)-\eta_1L(\eta L-1)\eta L-2\eta^2L^2(\eta L-1))16(\tau_1L)$

$$+(\eta L(\eta L+1)\eta_2L)16(\tau_1L)-(\frac{\eta L(\eta L+1)(2\eta L+1)}{3})16(\tau_1L)-(2\eta_2\eta L^2-\eta L(\eta L+1))16(\tau_1L)$$

$$+(\eta L(\eta L+1)\eta_1L)16(\tau_1L)-(\frac{\eta L(\eta L+1)(2\eta L+1)}{3})16(\tau_1L)-(2\eta_1\eta L^2-\eta L(\eta L+1))16(\tau_1L)$$

$((\eta_1L-1)\eta_1L(\eta_2L-1)\eta_2L-\eta L(\eta L-1))16(4\tau_1^2L^2)$

$-(\frac{\eta_1 L(\eta_1 L+1)(2\eta_1 L+1)}{3}+(8\eta_1^2 L^2+4\eta_1 L)\tau_1 L)(\frac{\eta_2 L(\eta_2 L+1)(2\eta_2 L+1)}{3}+(8\eta_2^2 L^2+4\eta_2 L)\tau_2 L)$
Let η be the minimum of η_1, η_2. Looking for the terms with no L's after dividing by L^4, we get:
$(32\eta_2\eta^2+32\eta_1\eta^2-32\eta^3)\tau_1+64\eta^2\tau_1^2$

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References

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