Computing Isotopy Types of Zero Sets of Circuit Polynomials

Texas A&M University REU 2023

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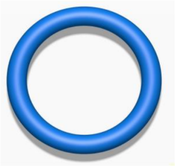
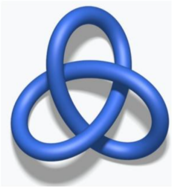
Introduction

- **Definition.** An *(ambient in $\mathbb{R}^n$) isotopy* between X,Y $\subseteq \mathbb{R}^n$ is a continuous function H:[0,1] x $\mathbb{R}^n \to \mathbb{R}^n$ satisfying:
  1. $H_T : \mathbb{R}^n \to \mathbb{R}$ defined by $H_T(x) = H(y,x)$ for all x $\in \mathbb{R}^n$, is a homeomorphism for each T $\in$ [0,1].
  2. H(0,X)=X for all x $\in \mathbb{R}^n$
  3. H(1,X)=Y





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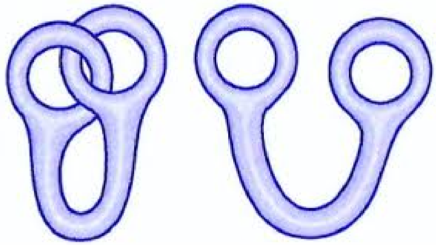

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Polytopes

- **Definition.** A polytope (in $\mathbb{R}^n$) is the convex hull of any finite subset of $\mathbb{R}^n$.

Figure 1: Point

Figure 2: Line Segment

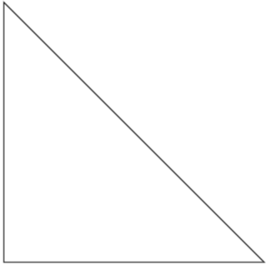


Figure 3: Triangle

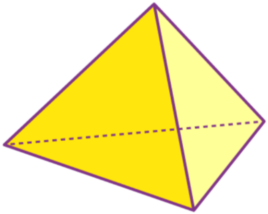
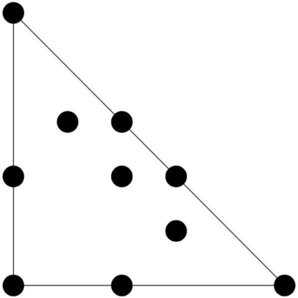


Figure 4: Pyramid

- **Definition:** We call $\{a_1, ..., a_T\} \subset \mathbb{R}^n$ a **circuit** iff rank of $\begin{bmatrix} 1......1 \\ a_1......a_T \end{bmatrix}$ is T-1.
  - Ex: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has the Rank 2=3-1
- **Newton polygons** are formed when you take the convex hull of exponent vectors.

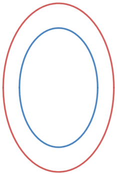


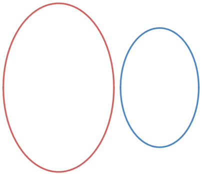
Hilbert's 16th Problem

- posed by David Hilbert in 1900
- Harnack (1876) investigated algebraic curves in the real projective plane and found that curves of degree n could have no more than

$$\frac{n^2 - 3n + 4}{2}$$

  connected components (pieces).
- **M-curves:** curves with maximally many ovals
- **Disposition** of ovals tells you the isotopy type.





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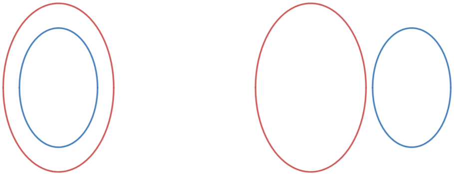

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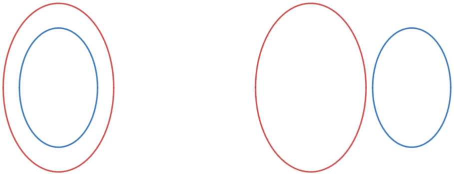

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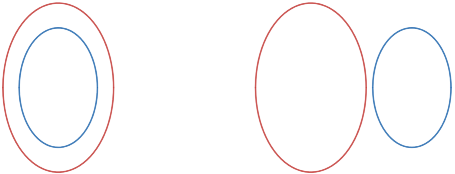
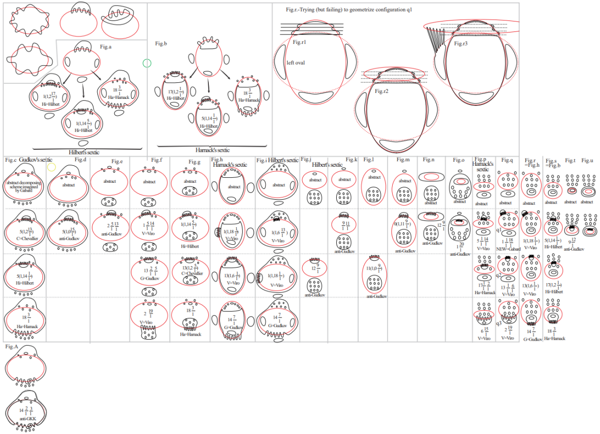


Hilbert's 16th Problem

The disposition of ovals is still unknown at n=8.



- Viro's patchworking method helps classify curves and surfaces.
- **Main Idea:** decompose a real algebraic variety into parts called patches, which are easier to analyze.

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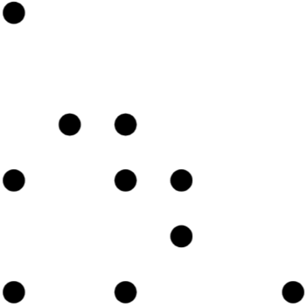

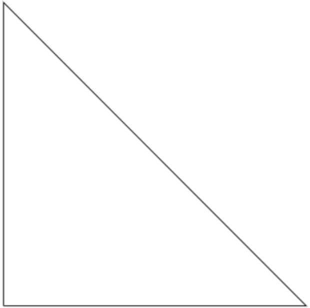

An Example of a Viro Diagram

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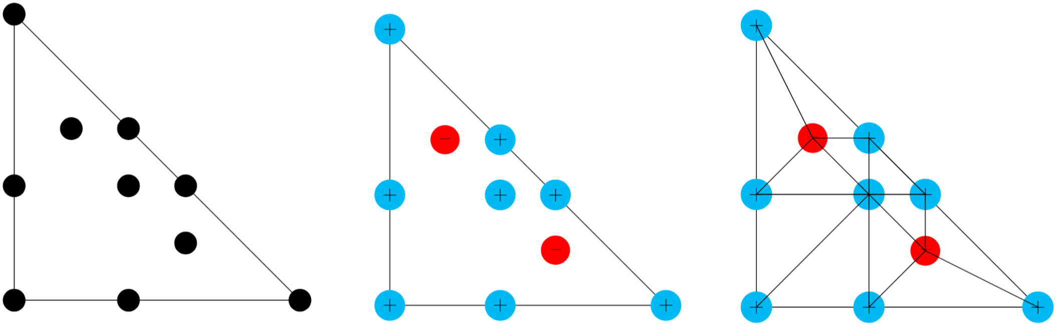

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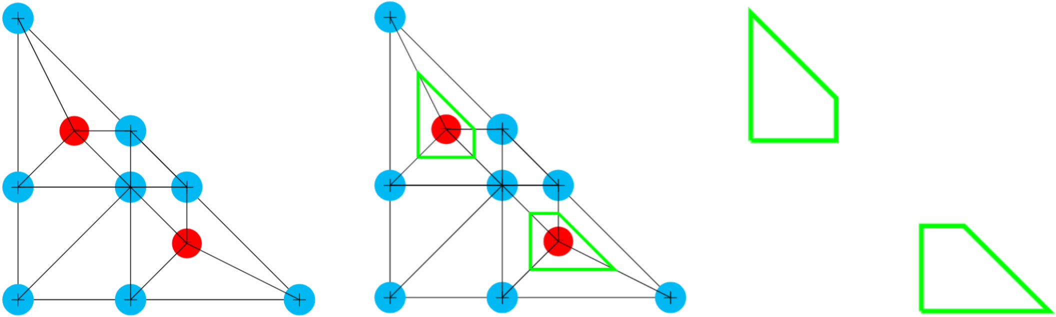

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Outline of Algorithm for Computing Isotopy Types

Input: $A := [a_1, \ldots, a_{n+2}] \in \mathbb{Z}^{nx(n+2)}$ with $\hat{A} := \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+2} \end{bmatrix}$ has rank n+1, and $c_1, \ldots, c_{n+2} \in \mathbb{R}$.

Output: A quadratic polynomial q with $Z_{\mathbb{R}}(q)$ isotopic to $Z_+(f)$, where $f(x) := \sum_{i=1}^{n+2} c_i x^{a_i}$

1. If $\text{sign}(c_1) = \cdots = \text{sign}(c_{n+2})$ then output $Z_+(f) = \emptyset$ and stop.
2. Let $b \in \mathbb{Z}^{(n+2)\times 1}$ be any generator for right nullspace of $\hat{A}$. If Sign(c)$\neq \pm$ Sign(b) then $Z_+(f)$ is isotopic to a hyperplane.
3. (Roughly) Compute the oriented matroid structure of A and compute sign of A-discriminant to obtain q.

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- Step 1: Are all coefficient signs the same?
  - E.g., $1+x+y+xy$ has an empty positive zero set because a sum of positive numbers is never 0!

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  - E.g., -1+x+y+xy

$$\hat{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad b\text{-vector} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

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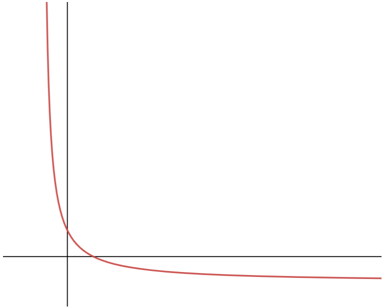

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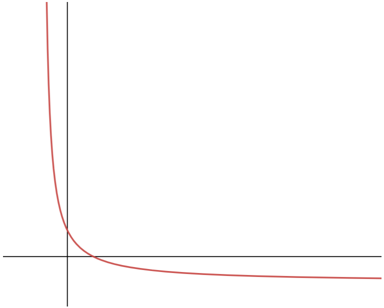

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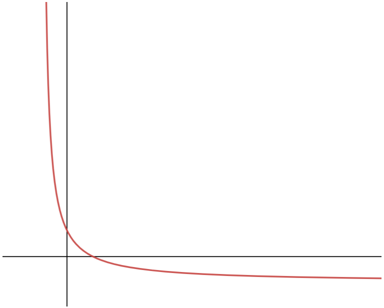

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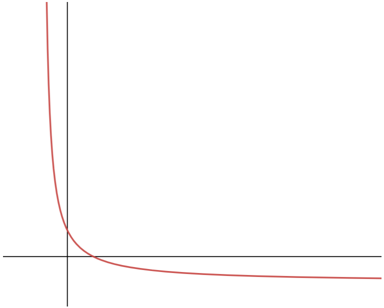

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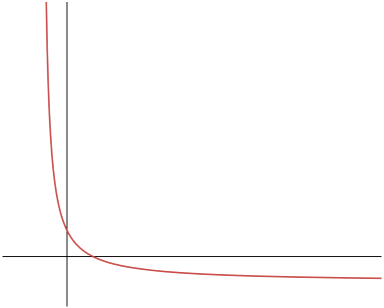

The Discriminate

Quadratic Discriminant. If a, b, c are real numbers, then $f(x) := c + bx + ax^2$ has 0, 1, or 2 real roots, according to the

$$\textbf{discriminant } \Delta_{\{0,1,2\}}(f) := b^2 - 4ac \text{ is } < 0, = 0, \text{ or } > 0.$$

Trinomial Discriminant. If a, b, c are *positive* real numbers, then $f(x) := c - bx^{39} + ax^{2006}$ has 0, 1, or 2 positive roots, according to the

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Hardest Case

- Step 3: Oriented Matroids and Discriminants
  - E.g., 2-x-y+xy
  - The simplex has 3 edges and the point $(1,1)$, which lies on the positive (outside) side of the edge. So for our example, we get the sequence $+++$. This will determine an index ℓ yielding $q(x_1,x_2)=x_1^2+\cdots+x_\ell^2-x_{\ell+1}^2-\cdots-x_2^2+_3$.
  - So then ...
  - b-vector is $\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$ and discriminant is $(\frac{2}{1})^1(\frac{-1}{-1})^-1(\frac{-1}{-1})^-1(\frac{1}{1})^1-1>0$

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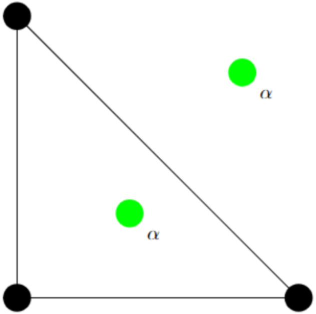
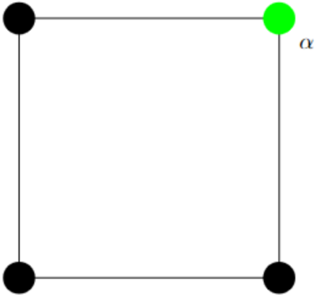
Shapes from Matroid Structure

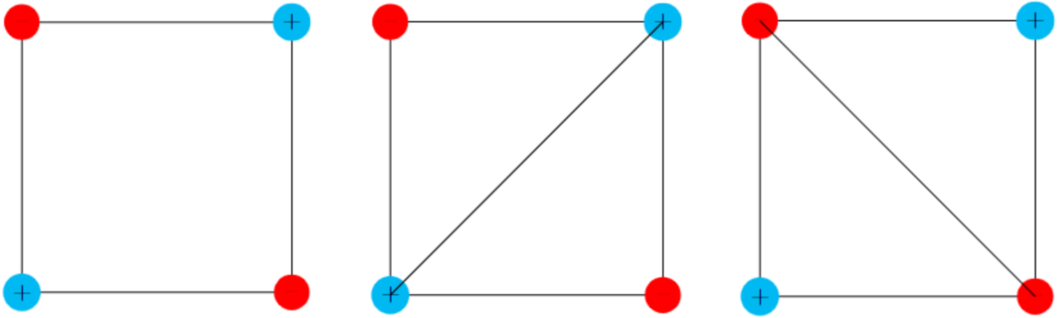
The position of the last support point α relative to the simplex determines what shapes can you get...





Viro Diagrams

2-x-y+xy
(x-1)(y-1)+1



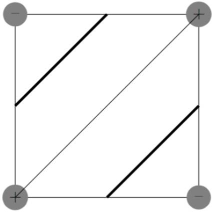


Figure 5: $\Delta \leq 0$

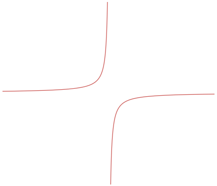


Figure 6: $\Delta \leq 0$

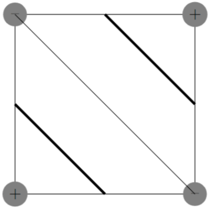


Figure 7: $\Delta > 0$

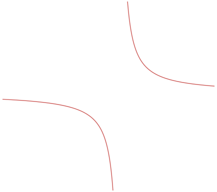


Figure 8: $\Delta > 0$

Triangulations

$$x^2 + y^2 + 1 - cxy$$

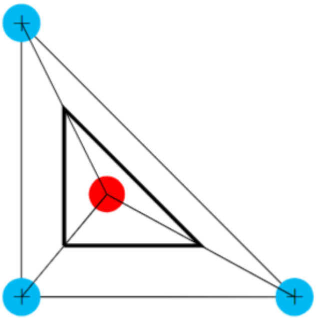


Figure 9: c>2.749459275

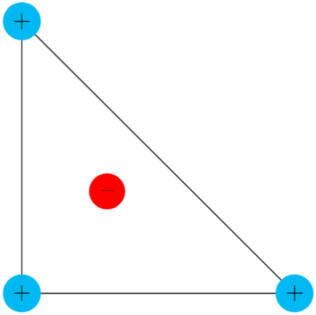


Figure 10: c=2.749459275

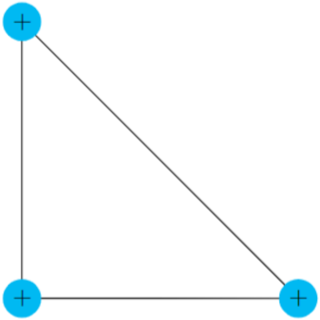


Figure 11: c<2.749459275

In Closing

What about higher dimensions?

In 1994, Gelfand, Kapranov, and Zelevinsky observed that for any circuit $\{a_1, \cdots, a_{n+2}\}$, provided $(\text{sign}(c_1), \cdots, \text{sign}(c_{n+2}))$ matches $\pm\,(\text{sign}(b_1), \cdots, \text{sign}(b_{n+2}))$ then the isotopy type of $Z_+(f)$ is determined by the sign of

$$\boxed{\Xi_A := \prod_{i=1}^{n+2} \left(\frac{c_i}{b_i}\right)^{b_i} - 1} \ldots$$

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In Closing

The Algorithm:

1. If $\text{sign}(c_1) = \cdots = \text{sign}(c_{n+2})$ then output $Z_+(f) = \emptyset$ and stop.
2. Let $b \in \mathbb{Z}^{(n+2)\times 1}$ be any generator for right nullspace of $\hat{A}$. If Sign(c)$\neq \pm$ Sign(b) then $Z_+(f)$ is isotopic to a hyperplane.
3. (Roughly) Compute the oriented matroid structure of A and compute sign of A-discriminant to obtain q.

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Thank You