Computing the Isotopy Type of the Positive Zero Set of Polys on a Circuit

Faith Ellison

Abstract

Understanding real curves is more subtle than understanding complex curves, as one can see from the literature on Hilbert's 16th Problem. This includes the determination of isotopy types, which means counting ovals and determining their possible dispositions. Newton polygons and amoebas are two tools that arose from studying complex curves and we'll see ways to refine these tools to tell us about real curves. In particular, we'll see an efficient algorithm to determine the isotopy type of the positive zero set of any bivariate tetranomial, including the high degree case. As a consequence, we'll see how it's better to use explicit quadratic forms instead of Viro diagrams to determine isotopy type.

1 Introduction

There are many tools by which one can understand the geometry of complex curves. We will start by considering amoebas and newton polygons, which both illustrate the deep connections between polynomials and polyhedra. Amoebas are one such tool, useful for getting rough metric information and understanding behavior of zero sets near infinity. For this research, we'll focus on polynomials in n variables having n+2 monomial terms.

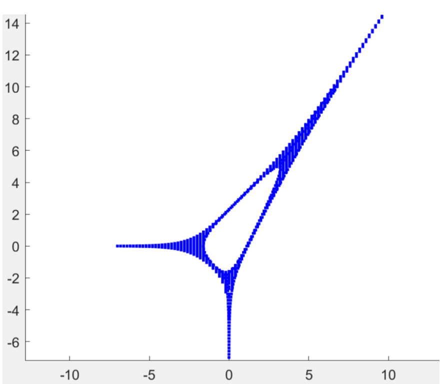

Figure 1: Amoeba$(1 + x^3 + y^2 - 10xy)$

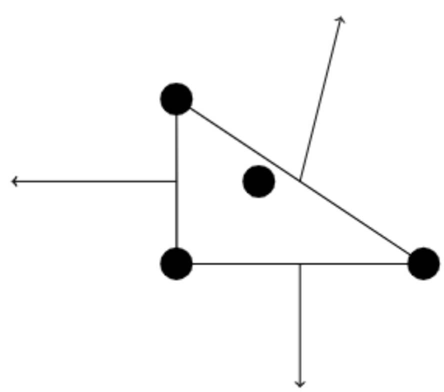

Figure 2: Newton Polygon Example

Amoebas were first studied in the 1990s by Gelfand, Kapranov, and Zelevinsky ([1]). In particular, the amoeba of a polynomial f is simply the image of the complex roots of f under the coordinate-wise absolute value map. The amoeba of $1 + x^3 + y^2 - 10xy$ can be seen in 1. The name "amoeba" comes from the mathematical amoeba having a synonymous shape to a biological amoeba. Furthermore, the direction of the amoeba's tentacles can be predicted by a polytope, which can be associated to the underlying polynomial. For example, in our current example, we simply take the convex hull of the exponent vectors $(0, 0)$, $(3, 0)$, $(0, 2)$, and $(1, 1)$ and obtain the triangle P with vertex $\{(0, 0), (3, 0), (0, 2)\}$.

Notice that the directions of the tentacles are parallel to the outer normal rays of P. P is our first example of a *Newton polygon*.

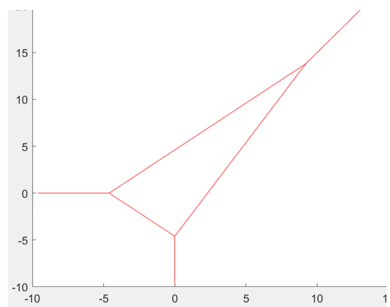

Figure 3: $\mathrm{ArchTrop}(1 + x^3 + y^2 - 10xy)$

Tropical varieties have their origin in work of Imre Simon from the 1960s on the algebra max-plus algebra over $\mathbb{R}$. Tropical varieties are piece-wise linear manifolds that approximate complex zero sets. Their precise construction ultimately involves working with fields like $\mathbb{C}$ and $\mathbb{C}\{\{t\}\}$ (the latter being the field of *Puiseux series* over $\mathbb{C}$), which are fields one can endow with a *valuation map*. A more detailed definition is given in (2.2). One type of tropical variety (known as an *Archimedean* tropical variety) can be obtained by approximating the zero set of a polynomial f with zero sets of binomials obtained from pairs of monomials of f. 6

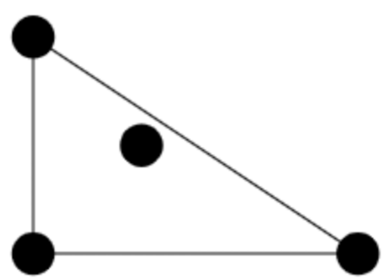

Figure 4: $\mathrm{Newt}(\mathrm{poly}(1 + \mathrm{x}^3 + \mathrm{y}^2 - 10\mathrm{xy}))$

Newton polygons date back to the work of Isaac Newton in 1676 where he used a simple diagrams to study branches of algebraic functions defined by bivariate polynomials.

Real zero sets and, specifically, positive zero sets of polynomials, are of fundamental importance in computational biology: Studying equilibria in biochemical reactions leads immediately to the zero sets of systems of polynomial equations, and the solutions actually give information about concentrations of reactants. So positive solutions are of the greatest importance.
Our main goal is to advance algorithmic solutions of polynomial systems by thoroughly understanding important special cases. In particular, we'll look at how to compute the isotopy types of positive zero sets of arbitrary degrees.

2 Background

In this section, we will discuss the background needed for the foundation of this research.

2.1 Amoebas

An approximation that resembles a biological amoeba on a graph. In complex analysis, a logarithmic map yields a new coordinate system. The amoeba is essentially the image of the original curve under the logarithmic coordinate transformation.

$$Amoeba(f) := \{(log|x|, log|y|) | f(x, y) = 0; x, y \in \mathbb{C}^*\}$$

Figure 5: Example Amoeba

2.2 Tropical Varieties

An algebraic variation stemming from tropical geometry where polynomial graphs resemble piece-wise linear networks. Where the function f is a Laurent Polynomial and the results produce subsets that exist in $\mathbb{R}$.[6,7]

$Archtrop(f) := \{w \in \mathbb{R} | max_i |c_i e^{a_i \cdot w}|$ is attained at two or more distinct indices i$\}$

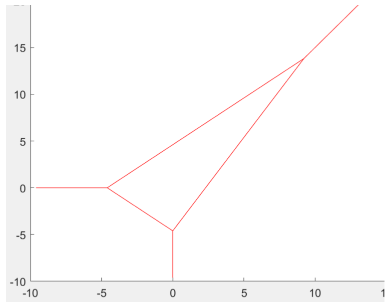

Figure 6: ArchTrop$(1 + x^3 + y^2 - 10xy)$

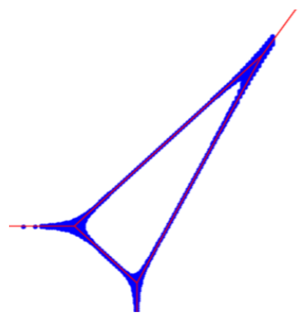

Figure 7: Example Tropical Variety lying on Amoeba

2.3 Newton Polygons

Definition 1. A tool to understand the behavior of polynomials that may undergo power series in the non-archimedean plane,

$$f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{R}$$

where a is some non-zero constant and n is an integer.

Newton polygons date back to work of Newton in the 17th century and, already then, he found that simple diagrams can help understand complicated polynomials. For polynomials in one variable, the Newton polygon in the Cartesian plane easily gives information about the number of complex roots of a polynomial: Length(Newt(f)) is exactly the number of nonzero roots of $\mathbb{C}$, counting multiplicity.

For polynomials in arbitrarily many variables, the Newton polygon gives information about how the complex roots of a polynomial behave at infinity: When the number of variables is exactly the dimension of the Newton polygon, the outer normal cones of the vertices of $Newt(f)$ predict exactly which way the unbounded connected components of the complement $\mathbb{R}^n Amoeba(f)$ tend to infinity. In two variables, you can also interpret this as the outer edge normals giving the directions of the tentacles of Amoeba(f).

The Archimedean Newton polygon (with respect to the valuation $log|.|$) gives rough metric information about the location of the coordinate-wise log-absolute values of the complex roots of a polynomial. There are explicit Hausdorff distance estimates between ArchTrop(f) and Amoeba(f), which are covered in [2].

Theorem 1. (Kapranov's Non-Archimedean Theorem) For any f in $\mathbb{C}\langle\langle s\rangle\rangle[x_1^{\pm1}, \cdots, x_n^{\pm1}$, we have $v_s(Z_s^*(f))$=$\text{Trop}_s(f) \cap v_s(\mathbb{C}\langle\langle s\rangle\rangle)^n$ [2].

Over non-Archimedean fields, the analogues of ArchTrop(f) and Amoeba(f) agree exactly. This is known as Kapranov's Non-Archimedean Amoeba Theorem. So when the underlying field is $\mathbb{C}\{\{t\}\}$, the analogue of ArchTrop(f) is called $Trop_t(f)$ and is defined slightly differently. Likewise, the analogue of Amoeba(f) is instead $ord_t(Z(f))$, where Z(f) denotes the zero set of f in $(\mathbb{C}\{\{t\}\})^n$ and ord_t denote the t-adic valuation on $\mathbb{C}\{\{t\}\}$.

3 Puisex Series

Around 1676, Newton was the first to find a connection between polygons and power series expansions for the graphs of curves, using fractional exponents. These power series expansions were later referred to as Puiseux series expansions, following Victor Puiseux's work around 1850. In the early 2000s, several researchers including Baker, Kapranov, Litvinov, Markwig, McClagan, Mikhalkin, Payne, and Sturmfels established the foundations of tropical algebraic geometry. It was around this time that Newton and Puiseux's work was reinterpreted as a way to compute tropical varieties over the field $\mathbb{C}\{\{t\}\}$ (the field of Puiseux series over $\mathbb{C}$ in one variable). $\mathbb{C}\{\{t\}\}$ is a non-Archimedean field.

Theorem

Theorem 2. (GKZ Amoeba Theorem) For any f in $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ with n-dimensional Newton polytope P, its amoeba complement $X := \mathbb{R}^n \setminus \text{Amoeba}(f)$ is a finite union of convex sets.

This theorem is significant because it gives a direct correspondence between Newton polytopes and the geometry of complex zero sets, i.e., a serious connection between polytopes and polynomials. In particular, the unbounded connected components of X are in bijective correspondence with the vertices of P, and the number of bounded components of X is at most the number of lattice points in the interior of P. Furthermore, each unbounded connected component of X contains a translation of the outer normal cone of a unique vertex of P.

The part on unbounded components can be proved via complex analysis, by relating each unbounded component with the convergence region of a (multivariate) Laurent series expansion of 1/f. The portion on bounded connected components can be proved by applying metric estimates on ArchTrop(f) [2].

4 Hilbert's 16th Problem

Investigates algebraic curves in the real projective plane. Used to solve the first situation regarding Hilbert's 16th Problem.

Theorem 3 (Harnack's Curve Theorem)**.** Harnack found that curves of degree n could have no more than

$$\frac{n^2 - 3n + 4}{2}$$

separate connected components. He concluded that M-curves are curves with n+1 ovals [3].

Theorem 4 (Viro's Theorem)**.** Also known as Viro's Patchworking Method is a theory that is used to classify curves and spaces. Instead of the polynomial forming the patches, the patches themselves outline the polynomial.

5 Results

Understanding how to take a polynomial through the steps of the algorithm can be tedious. To understand my findings I will be using the polynomial $2 - x - y + xy$ as an example of how the algorithm works.

Assume $a_1 ... a_[n+2] \in \mathbb{Z}^{nx(n+2)}$ with $\hat{A} := \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+2} \end{bmatrix}$ has a rank of n+1, and $c_1, \cdots, c_{n+2} \in \mathbb{R}$.

The following steps will produce a quadratic polynomial q with $Z_{\mathbb{R}}(q)$ isotopic to $Z_+(f)$, where $f(x) := \Sigma_{i=1}^{n+2} c_i z^{a_i}$

1. The signs of the coefficients determine the first basis of the algorithm being if there is a positive zero set or an empty positive zero set. If sign(c_1)=$\cdots$=sign(c_2) then the output $Z_+(f) = \emptyset$, otherwise continue to the next step.

2. Let b-vector $\in \mathbb{Z}^{(n+2)x1}$ be any generator for the right nullspace of $\hat{A}$. If the Sign(c) $\neq \pm$ Sign(b) then the $Z_+(f)$is isotopic to a hyperplane. Otherwise, continue to the final step.

3. Compute the matroid structure of A and compute the sign of the A-discriminant to obtain q.

For our example, $2 - x - y + xy$, (1) the sign of the coefficients are not equal (+−+). Therefore, we move forward comparing the compatibility of b and c (2).

$\hat{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b-vector= $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ The $\pm$sign(b) are +−+ and -++-, which does equal the sign(c) (+−+) so to solve for the isotopy of the positive zero set you must compute the discriminant and matroid structure (3).

Definition

Definition 2. (Quadratic Discriminant)

If a,b,c are real numbers, then f has 0, 1, or 2 real roots according to the discriminant $\Delta_{0,1,2}(f) := b^2 - 4ac$ *is* $<,>$, *or* $= 0$

Using substitution, the discriminant Δ of $2-x-y+xy$ is > 0. Moreover, when observing your matroid structure and its triangulation, you will get an accurate depiction of the $Z_+(f)$. Using Viro diagrams allows you to see the isotopy type and positive zero set as you can see in the following figures.

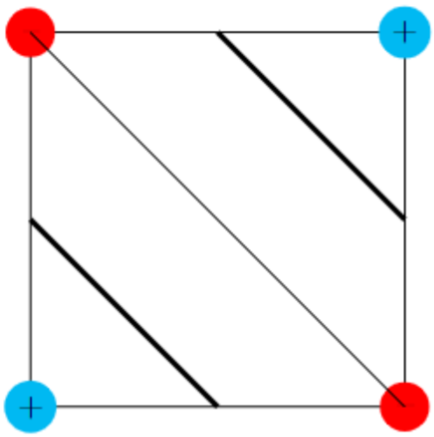

Figure 8: $\Delta \geq 0$

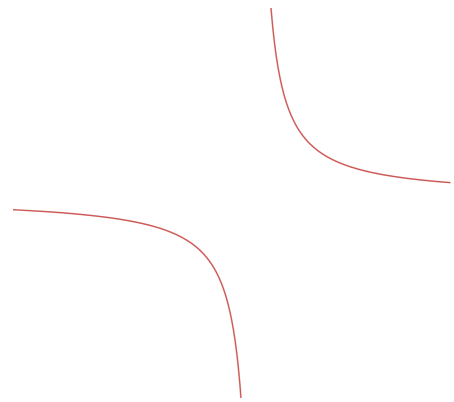

Figure 9: $\Delta > 0$

This algorithm does work for arbitrarily higher degrees. The hardest cases do require the computation of the matroid structure and the discriminant, while the easier cases require mere observation of signage.

6 Acknowledgements

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