Representations of Fermionic Modular Categories from Topological Quantum Computing

UNDERGRADUATE RESEARCH SCHOLAR

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Background

Quantum computing offers benefits over traditional bit-arithmetic computing due to the use of Qubits, which provide more computational power by "using the two characteristic attributes of quantum mechanics – superposition and entanglement"[1]. However, quantum particles are highly prone to environmental interference. Topological Quantum Computing is a method of quantum computation which focuses on encoding information in topological invariants. By using topological symmetries, interference can be mitigated since the general structure, not distance and angle degree, decides equivalence in topological structures.

Project Overview:

- Generate S and T matrix representations
- Investigate resulting matrix group using magma algebraic software[2]
- Characterize and categorize resulting group from gathered data

Methods

The matrix representation of FMCs and MMCs, are generated by the use of an S and T matrix which take on a unique form for each category. [3] supplies the I_{sing} theory containing the desired Fermion which is tensored with some base group:

$$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

Fermionic Modular Categories

- Structure is based on the use of Z_N cyclic groups, an example provided in [3]
- Derived from the Kronecker product of the I_{sing} theory with the objects of the FMC
- Reduced form used based on entries commuting in:

$$S_{(x,y)} = \sigma_{(x,y)} \tag{1}$$

- Z_K where $K = 8n$ for $n \in \mathbb{Z}^+$, always contains a non-trivial Boson
- Results in square matrices of dimension $\frac{3}{4}K$

The S matrix has structure:

$$\begin{bmatrix} A & \sqrt{2}B \\ \sqrt{2}B^T & C \end{bmatrix} \tag{2}$$

where:

$$\mathrm{A}_{a,b} = e^{i\frac{4\pi}{2K}(2a*2b)} \qquad 0 \le a, b < \frac{2}{4}K \tag{3}$$

$$\mathrm{B}_{c,d} = e^{i\frac{4\pi}{2K}(2c(2d+1))} \qquad \frac{2}{4}K \le c, d < \frac{3}{4}K \tag{4}$$

and **C** a zero square matrix of dimension $\frac{1}{4}K$

The T matrix is a diagonal square matrix of dimension $\frac{3}{4}K$ with entries:

$$e^{\frac{\pi i}{8}}(\theta_0, \theta_2, \ldots, \theta_{2a}, \theta_1, \theta_3, \ldots, \theta_{2j+1}) \tag{5}$$

Where $\theta_a = e^{\frac{i\pi a^2}{K}}$ for $0 \le a \le K-1$ and $\theta_{2j+1} = e^{\frac{i\pi(2j+1)^2}{n}}$ for $0 \le j \le \frac{K}{2}$

Metaplectic Modular Categories

- Structure is based on Type B $SO(m)_2$ categories as described in [4]
- For $SO(m)_2$ where $m = 2r+1$ for $r \in \mathbb{Z}^+$, S Matrix is square matrix of dimension $r+4$
- The generalized S matrix structure is:

$$\frac{1}{\sqrt{2m}}\begin{bmatrix} 1 & 1 & \gamma_1 & \gamma_2 & \ldots & \gamma_r & \sqrt{m} & \sqrt{m} \\ 1 & 1 & \gamma_1 & \gamma_2 & \ldots & \gamma_r & -\sqrt{m} & -\sqrt{m} \\ \gamma_1 & \gamma_1 & 4\cos(\frac{a_1 b_1 \pi}{m}) & 4\cos(\frac{a_1 b_2 \pi}{m}) & \ldots & \cos(\frac{a_1 b_r \pi}{m}) & 0 & 0 \\ \gamma_2 & \gamma_2 & 4\cos(\frac{a_2 b_1 \pi}{m}) & 4\cos(\frac{a_2 b_2 \pi}{m}) & \ldots & \cos(\frac{a_2 b_r \pi}{m}) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ \gamma_r & \gamma_r & 4\cos(\frac{a_r b_1 \pi}{m}) & 4\cos(\frac{a_r b_2 \pi}{m}) & \ldots & 4\cos(\frac{a_r b_r \pi}{m}) & 0 & 0 \\ \sqrt{m} & -\sqrt{m} & 0 & 0 & \ldots & 0 & \sqrt{m} & -\sqrt{m} \\ \sqrt{m} & -\sqrt{m} & 0 & 0 & \ldots & 0 & -\sqrt{m} & \sqrt{m} \end{bmatrix}$$

where $\gamma_1 = \cdots = \gamma_r = 2$ and $a_i = i, b_j = j$

- The T matrix is a diagonal square matrix of dimension $r+4$ with entries:

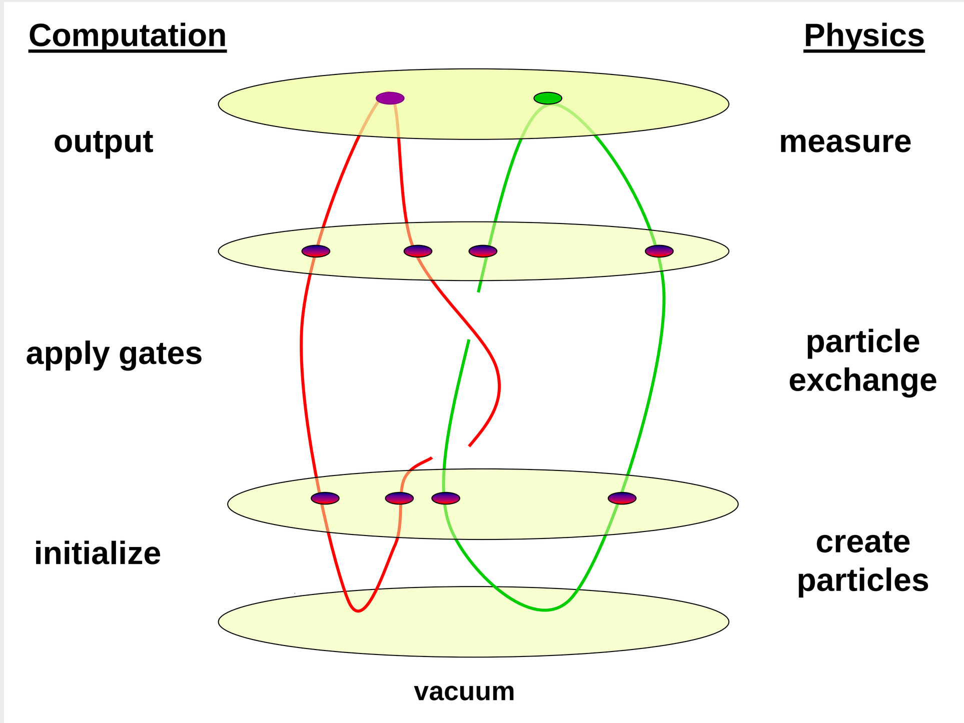
$$(1, 1, \theta_{\gamma_1}, \theta_{\gamma_2}, \ldots, \theta_{\gamma_r}, e^{\frac{2\pi i}{8}}, e^{\frac{2\pi i}{8}}) \tag{6}$$

Where $\theta_{\gamma_j} = e^{-j^2\frac{2\pi i}{m}}$

Forward Work

- Improve the Code
- Investigate other examples from [3]

Computation Model



http://www.math.tamu.edu/ rowell/RowellTyler09nopause.pdf

Computation

Ex: Fermionic Modular Category algorithm:

procedure GENERATE FMC MATRIX GROUP
  Input: N a positive integer
  $K = 8 * N$
  ZField ← Cyclotomic Field with Kth root of unity
  Set root to ZField's root element
A Matrix:
  A is Zero square matrix of dimension $\frac{3}{4}K$
  for i in $[0,\frac{K}{2})$ **do**
    for j in $[0,\frac{K}{2})$ **do**
      $A[i,j] = \mathrm{root}^{2i*2j}$
B Matrix:
  B is Zero square matrix of dimension $\frac{3}{4}K$
  for i in $[0,\frac{K}{2})$ **do**
    for j in $[\frac{K}{2},\frac{3}{4}K)$ **do**
      $B[i,j] = \sqrt{2}\,\mathrm{root}^{2i*(2j+1)}$
S Matrix:
  S is Zero square matrix of dimension $\frac{3}{4}K$
  S = A + B + Transpose of B
  Scale S by $\frac{1}{\sqrt{K}}$
T Matrix:
  T is Zero square matrix of dimension $\frac{3}{4}K$
  for i in $[0,\frac{K}{2})$ **do**
    $T[i,i] = \mathrm{root}^{2*i^2}$
  for j in $[\frac{K}{2},\frac{3}{4}K)$ **do**
    $T[j,j] = \mathrm{root}^{(2j+1)^2+8}$
Generate Group:
  A ← Matrix Group generated from S and T^2

Results

Fermionic Modular Category groups

N	Z	Quotient Chain	Order(Factored)	Order	Solvable	Nilpotent	Special
1	8	32, 8, 4, 1	2^5	32	False	True	True
2	16	256, 16, 8, 4, 1	2^8	256	True	True	False
3	24	18432, 1152, 576, 288	$2^{11} \cdot 3^2$	18432	True	False	—–
4	32	2048, 64, 16, 8, 4, 1	2^{11}	2048	True	True	False
5	40	921600*	$2^{12} \cdot 3^2 \cdot 5^2$	921600	—–	False	—–
6	48	147456, 2304, 1152, 576, 288	$2^{14} \cdot 3^2$	147456	True	False	—–
7	56	7225344*	$2^{14} \cdot 3^2 \cdot 7^2$	7225344	—–	—–	—–
8	64	65536*	2^{16}	65536	True	—–	False
9	72	13436928*	$2^{11} \cdot 3^8$	13436928	True	False	—–

Metaplectic Modular Category groups

r	SO(n)	Z	Quotient Chain	Order(Factored)	Order	Final Simple	Solvable	Simple
1	3	24	768, 96, 48, 12	$2^8 \cdot 3^1$	768	True	True	False
2	5	40	1920, 480, 240, 60	$2^7 \cdot 3^1 \cdot 5^1$	1920	True	False	False
3	7	56	10752, 1344, 672, 168	$2^9 \cdot 3^1 \cdot 7^1$	10752	True	False	False
4	9	72	10368, 2592, 1296, 324	$2^7 \cdot 3^4$	10368	False	True	False
5	11	88	42240, 5280, 2640, 660	$2^8 \cdot 3^1 \cdot 5^1 \cdot 11^1$	42240	True	False	False
6	13	104	34944, 8736, 4368, 1092	$2^{11} \cdot 3^2 \cdot 5^1$	34944	True	False	False
7	15	120	92160, 5760, 2880, 720	$2^{11} \cdot 3^2 \cdot 5^1$	92160	False	False	False
8	17	136	78336, 19584, 9792, 2448	$2^9 \cdot 3^2 \cdot 17^1$	78336	True	False	False
9	19	152	218880*	$2^8 \cdot 3^2 \cdot 5^1 \cdot 19^1$	218880	—–*	False	False

*denotes projected result due to computational limitations

References

[1] Z. W. Eric C. Rowell, "Mathematics of topological quantum computing,"

[2] W. Bosma, J. Cannon, and C. Playoust, "The Magma algebra system. I. The user language," *J. Symbolic Comput.*, vol. 24, no. 3-4, pp. 235–265, 1997. Computational algebra and number theory (London, 1993).

[3] P. H. Bonderson, "Non-abelian anyons and interferometry,"

[4] D. Naidu and E. C. Rowell, "A finiteness property for braided fusion categories," 2009.