Monomial Solutions to Generalized Yang-Baxter Equations in Low Dimensions

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Yang-Baxter Equations

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$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where I is the identity matrix and $\otimes$ is the Kronecker product.

Generalized Yang-Baxter Equations

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Definition

- If V is a complex vector space of dimension d, the (d, m, ℓ)-gYBE is an equation for an invertible operator $R : V^{\otimes m} \to V^{\otimes m}$ such that
$$\left(R \otimes I_V^{\otimes \ell}\right)\left(I_V^{\otimes \ell} \otimes R\right)\left(R \otimes I_V^{\otimes \ell}\right) = \left(I_V^{\otimes \ell} \otimes R\right)\left(R \otimes I_V^{\otimes \ell}\right)\left(R \otimes I_V^{\otimes \ell}\right).$$

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- The d-dimensional YBE corresponds to the $(d, 2, 1)$-gYBE.
- $R_\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^{-1} & 0 & -\zeta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & \zeta & 0 & 0 & 0 & 0 \\ \zeta & 0 & \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta^{-1} & 0 & \zeta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^{-1} & 0 & -\zeta^{-1} \\ 0 & 0 & 0 & 0 & -\zeta^{-1} & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta & 0 & \zeta \end{pmatrix}$

  where $\zeta = e^{2\pi i/8}$.

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  where $\zeta = e^{2\pi i/8}$.
- In 2011, R. Chen used R_ζ to find three solution families of the $(2, 3, 1)$-gYBE.

Permutation Solutions

Remarks

- Due to the enormous number of equations that must be solved in order to completely solve a gYBE, we focused on the simplest set of solutions, the permutation solutions proposed by V. Drinfeld.

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Example

- There are 4 nontrivial permutation solutions to the 2-D YBE:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Generating Permutation Solutions

Remarks

- To find all of the permutation solutions to the $(2,3,1)$- and $(2,3,2)$-gYBE the most straight forward approach is to generate all 8! permutation matrices of order 8 and then plug them into the equations and see if they satisfy either one.

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- We implemented Heap's algorithm for generating permutations in *Maple* so that it treats the columns of the matrix it is operating on as the elements that are permuted.

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- We implemented Heap's algorithm for generating permutations in *Maple* so that it treats the columns of the matrix it is operating on as the elements that are permuted.

Results

- Using our process, we discovered 14 nontrivial solutions to the $(2, 3, 1)$-gYBE and 10 nontrivial solutions to the $(2, 3, 2)$-gYBE.

Generating Permutation Solutions

Example

- $R_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

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- This can be more succinctly described using the permutation $(0, 6, 5, 3)(1, 7, 4, 2)$.

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- $Q := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$, $R := AQ = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \\ 0 & 0 & d & 0 \end{pmatrix}$

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- $d^2(a-b), db(c-d), db(a-b), -c(ad-bc), ca(c-d), b^2(c-d), -a(ad-bc), a^2(c-d), d(a-b)(a^2+ab+b^2)$

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- $R = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & a \\ c & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$

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- Now checking for restrictions so that R is unitary: $a\bar{a} = 1, c\bar{c} = 1$
- $R = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & a \\ c & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}, |a| = |c| = 1$

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Remark

- We want to find Boolean functions f, g, h such that $|a, b, c> \mapsto |f(a, b, c), g(a, b, c), h(a, b, c)>$.
- It turns out that we can describe all $(2, 3, 1)$- and $(2, 3, 2)$-gYBE permutation solutions in this form using only negation and XOR operations.

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- To illustrate what we mean, we will consider $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

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- We make a truth table that describes the permutation:

a	b	$f(a,b)$	$g(a,b)$
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- From the table we find that the Boolean representation of this solution is $|a, b> \mapsto |\overline{b}, a>$.

Boolean Representation of Solutions: Nontrivial $(2, 3, 1)$-gYBE Solutions

Results

- $R_{01} : |a, b, c > \mapsto |\overline{a \oplus b}, b, \overline{b \oplus c} >$
- $R_{02} : |a, b, c > \mapsto |b, a, c >$
- $R_{03} : |a, b, c > \mapsto |a, c, b >$
- $R_{04} : |a, b, c > \mapsto |a, a \oplus b \oplus c, c >$
- $R_{05} : |a, b, c > \mapsto |\overline{b}, \overline{a}, c >$
- $R_{06} : |a, b, c > \mapsto |a \oplus b, \overline{b}, \overline{b \oplus c} >$
- $R_{07} : |a, b, c > \mapsto |a \oplus b, b, b \oplus c >$

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- $R_{08} : |a, b, c> \mapsto |\overline{b}, a, c>$
- $R_{09} : |a, b, c> \mapsto |a, \overline{c}, \overline{b}>$
- $R_{10} : |a, b, c> \mapsto |\overline{a \oplus b}, \overline{b}, b \oplus c>$
- $R_{11} : |a, b, c> \mapsto |a, \overline{a \oplus b \oplus c}, c>$
- $R_{12} : |a, b, c> \mapsto |b, \overline{a}, c>$
- $R_{13} : |a, b, c> \mapsto |a, \overline{c}, b>$
- $R_{14} : |a, b, c> \mapsto |a, c, \overline{b}>$

Boolean Representation of Solutions: Nontrivial $(2,3,2)$-gYBE Solutions

Results

- $S_{01} : |a, b, c> \mapsto |c, b, a>$
- $S_{02} : |a, b, c> \mapsto |\overline{b \oplus c}, b, \overline{a \oplus b}>$
- $S_{03} : |a, b, c> \mapsto |c, a \oplus b \oplus c, a>$
- $S_{04} : |a, b, c> \mapsto |\bar{c}, \overline{a \oplus b \oplus c}, \bar{a}>$
- $S_{05} : |a, b, c> \mapsto |b \oplus c, b, a \oplus b>$
- $S_{06} : |a, b, c> \mapsto |\bar{c}, b, \bar{a}>$
- $S_{07} : |a, b, c> \mapsto |\bar{c}, b, a>$
- $S_{08} : |a, b, c> \mapsto |b \oplus c, b, \overline{a \oplus b}>$
- $S_{09} : |a, b, c> \mapsto |\overline{b \oplus c}, b, a \oplus b>$
- $S_{10} : |a, b, c> \mapsto |c, b, \bar{a}>$

Extension to Larger Solutions: $(2,4,3)$-gYBE Solutions

Remark

- The boolean patterns also extend to (some) permutation $(2,4,2)$- and $(2,4,3)$-gYBE solutions.

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- The $(2,4,2)$-gYBE solutions are uninteresting, since they are identical to the $(4,2,1)$-gYBE solutions (4-D YBE solutions) which have been classified.

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Remark

- The boolean patterns also extend to (some) permutation $(2, 4, 2)$- and $(2, 4, 3)$-gYBE solutions.
- The $(2, 4, 2)$-gYBE solutions are uninteresting, since they are identical to the $(4, 2, 1)$-gYBE solutions (4-D YBE solutions) which have been classified.
- There are several new $(2, 4, 3)$ solutions:
  - $Y_{01} : |a, b, c, d > \mapsto |d, b, c, a >$
  - $Y_{02} : |a, b, c, d > \mapsto |b \oplus c \oplus d, b, c, a \oplus b \oplus c >$
  - $Y_{03} : |a, b, c, d > \mapsto |\overline{b \oplus c \oplus d}, b, c, \overline{a \oplus b \oplus c} >$
  - $Y_{04} : |a, b, c, d > \mapsto |\overline{b \oplus c \oplus d}, b, c, a \oplus b \oplus c >$
  - $Y_{05} : |a, b, c, d > \mapsto |b \oplus c \oplus d, b, c, \overline{a \oplus b \oplus c} >$

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$$R_X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Other Solutions: A New Solution

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- In addition to the permutation solutions to the $(2, 3, 2)$-gYBE, I also found a new solution resembling R_X:

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- In addition to the permutation solutions to the $(2, 3, 2)$-gYBE, I also found a new solution resembling R_X:

$$R_D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Results

- In addition to the permutation solutions to the $(2, 3, 2)$-gYBE, I also found a new solution resembling R_X:

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- This solution is not locally conjugate to R_X.

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- Is it possible to represent all 2 dimensional solutions using the Boolean representation presented here?
- Is there a similar representation for permutation solutions with dimension greater than 2?
- Are there more $(2,3,2)$-gYBE non-permutation solutions resembling R_X and R_D?

Questions?

Thank You!