Homework 7

Math 171H (section 201), Fall 2023

This homework is due on **Tuesday, October 10** at the start of class. (Turn in answers to questions 1–9.)

0. Read Sections 3.3–3.4

1. Evaluate the following limits. (No explanation required, but show your work.)

   (a) $\lim_{x\to 0} \frac{\sin x}{x}$

   (b) $\lim_{x\to 0} \frac{x}{\sin x}$

   (c) $\lim_{x\to \infty} \frac{\sin x}{x}$

   (d) $\lim_{x\to \pi} \frac{\sin x}{x}$

   (e) $\lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2}$

   (f) $\lim_{\theta\to 0} \frac{\sin^2\theta}{3\theta}$

2. Evaluate the following derivatives. (No explanation required, but show your work.)

   (a) $y = e^x \sin x$

   (b) $y = \frac{x\sin x}{1-\cos x}$

   (c) $f(\theta) = \cos^2\theta$

   (d) $f(\theta) = \tan^2(3\theta)$

   (e) $y = (3x-1)^2(2x+2)^{-4}3^x$

   (f) $y = \sqrt{1 + 1/\sqrt{2x}}$

   (g) $f(x) = x^2 \cdot g(1-x)$ (give your answer in terms of the functions g and g')

3. Give a formula for the derivative of $f(g(h(x)))$.

4. (a) Give examples of functions $f(x)$ and $g(x)$ for which $(f(x)g(x))' = f'(x)g'(x)$.

   (b) Give examples of functions $f(x)$ and $g(x)$ for which $(f(x)g(x))' \neq f'(x)g'(x)$.

5. Is it possible to write $f(x) = 2 + x$ as the product of two differentiable functions, $g(x)$ and $h(x)$, for which $g(0) = h(0) = 0$? Prove your answer. (*Hint*: Take a derivative.)

6. Consider the function $f(x) = a\cos x + b\sin x$, where a and b are real numbers. Show that $f^{(4)} = f$ (here, $f^{(4)}$ denotes the fourth derivative of f).

7. Assume that f is twice-differentiable everywhere (here, "everywhere" means on all of $\mathbb{R}$) and that:

$$\begin{aligned} f''(x) + f(x) &= 0 \\ f(0) = f'(0) &= 0 \end{aligned} \tag{1}$$

   (a) Multiply the first equation in (1) by $f'(x)$, and use the result to show that

$$\left((f')^2 + f^2\right)' \;=\; 0 \tag{2}$$

   (b) Use equation (2) to show that $f(x) = 0$. (You may use the fact [proven later this semester] that if $g' = 0$ then g is a constant function.)

8. Prove the following: *If f is twice-differentiable everywhere and*

$$\begin{aligned} f''(x) + f(x) &= 0 \\ f(0) &= a \\ f'(0) &= 0\ , \end{aligned}$$

   then $f(x) = a\cos x + b\sin x$.
   (Hint: Apply the previous problem to the function $h(x) = f(x) - a\cos x - b\sin x$.)

9. A polynomial $f(x)$ has a **double root** a if $f(x) = (x-a)^2 g(x)$, for some polynomial $g(x)$.

   (a) Prove that a is a double root of $f(x)$ if and only if a is a double root of $f(x)$ and a double root of $f'(x)$.

   (b) Describe the values of a, b, c, with $a \neq 0$, for which $f(x) = ax^2 + bx + c$ has a double root. What does such a parabola $y = f(x)$ look like?