Homework 10

Math 415 (section 502), Fall 2015

This homework is due on Thursday, November 5. You may cite results from class.

0. *(This problem is not to be turned in.)*
   (a) Read Section 15.
   (b) Section 15 #2, 4, 10, 36
1. Complete the following sentences. (No proofs necessary for this problem.)
   (a) The order of $6 + \langle 4 \rangle$ is the factor group $\mathbb{Z}_8/\langle 4 \rangle$ is ____________.
   (b) If H is a __________________ of a group G, then the __________________ form a group, denoted by G/H, in which the operation is defined by __________________ and the identity element is ____________.
   (c) A group G is *simple* if ____.
   (d) In the group $G = \mathbb{Q} \times \mathbb{Q}$, the elements $(-1, 3)$ and $(0, \_\_\_)$ are in the same coset of the subgroup $H := \{(x, y) \mid y = -2x\}$.
2. True/false. (No proofs necessary for this problem.)
   (a) For any group G, the set of homomorphisms $G \to G$ forms a group under composition.
   (b) Up to isomorphism, there is a unique abelian group of order 10.
   (c) $\mathbb{R}^+$ is a normal subgroup of $\mathbb{R}$.
   (d) The following is a well-defined function: $\phi : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}$ given by $\phi(n + 5\mathbb{Z}) := n$.
3. *Prove or disprove*: There exists a group G and a homomorphism $\phi : S_3 \to G$ for which the kernel is $\langle (12) \rangle$.
4. Assume that G and G' are finite groups of the same order ($|G| = |G'| < \infty$). Prove that the following are equivalent for a homomorphism $\phi : G \to G'$: (1) ϕ is an isomorphism, (2) ϕ is 1-1 (injective), and (3) ϕ is onto (surjective).
5. Let n be a positive integer. For $i = 1, 2, \ldots, n$, let $\phi_i : \mathbb{Z}_n \to \mathbb{Z}_n$ be the function given by $\phi_i(x) := ix$.
   (a) Prove the following: $\{\text{homomorphisms } \mathbb{Z}_n \to \mathbb{Z}_n\} = \{\phi_i \mid i = 1, 2, \ldots, n\}$.
   (b) Prove the following: $\text{Aut}(\mathbb{Z}_n) = \{\phi_i \mid \gcd(i, n) = 1\}$.
   (c) Prove that $|\text{Aut}(\mathbb{Z}_p)| = p - 1$ if p is a prime number.
   (d) Is $\text{Aut}(\mathbb{Z}_6)$ cyclic? Explain.

6. Let $\phi : G \to G'$ be a homomorphism. Assume that G has order 20.

   (a) Could $\ker(\phi)$ have order 6? Explain.

   (b) Prove that if $\ker(\phi)$ has order 4, then the order of G' (if finite) is at least 5.

7. (a) Which elements of $\mathbb{R}/\mathbb{Z}$ have finite order? Give a proof.

   (b) Which elements of $\mathbb{R}/\mathbb{Q}$ have finite order? Give a proof.

8. Assume that H and K are both normal subgroups of a group G and that $K \subset H$. On the previous homework, you proved that K is a *normal* subgroup of H.

   (a) Write down a function $\phi : G/K \to G/H$ for which the kernel is H/K. Prove that your function ϕ is well-defined, is a homomorphism, and has the correct kernel. Conclude that H/K is a *normal* subgroup of G/K.

   (b) Use the fundamental homomorphism theorem to prove that

$$(G/K)/(H/K) \quad \cong \quad G/H \, .$$

9. Section 15 #14, 19, 34