Homework 13

Math 415 (section 502), Fall 2015

This homework is due on TUESDAY, December 1. You may cite results from class or previous homeworks/exams.

0. *(This problem is not to be turned in.)*
   (a) Read Section 20.
   (b) Section 20 #7, 24
1. True/false (No proofs necessary for this problem.)
   (a) $\mathbb{Z}_{50}$ is an integral domain.
   (b) 0 is a zero divisor in every ring.
   (c) If a ring R has a zero divisor, then R is *not* a field.
   (d) The polynomial ring $\mathbb{Q}[x]$ is a field.
   (e) $\mathbb{Z}_{30}[x]$ is an integral domain.
   (f) The set of *continuous* functions $\mathbb{R} \to \mathbb{R}$ is a commutative ring with unity $1 \neq 0$.
   (g) The set of *constant* functions $\mathbb{R} \to \mathbb{R}$ forms a subring of the ring of functions $\mathbb{R} \to \mathbb{R}$.
2. Prove that if $\phi : R \to R'$ is a ring homomorphism, both rings R and R' have unity, and $\phi(1)$ is a unit, then $\phi(1) = 1$.
3. (a) Is $\mathbb{Z}_{12}^*$ (the group of units of $\mathbb{Z}_{12}$) cyclic? No proof necessary, but show your work. (You might look ahead and do #5b at this time.)
   (b) Prove that if $\varphi(n)$ is a prime number, then $\mathbb{Z}_n^*$ is cyclic. (Here, φ is the *Euler phi-function.*)
   (c) Is the converse of (b) true? Explain.
4. Let p and q be distinct prime numbers.
   (a) How many units does $\mathbb{Z}_{p^2}$ have? How many zero divisors? Give a proof.
   (b) How many units does $\mathbb{Z}_{pq}$ have? How many zero divisors? Give a proof.
5. (a) Prove that for $x \in \mathbb{Z}_n$, if $x^2 = 1$ (in $\mathbb{Z}_n$), then x is a unit.
   (b) Find all $x \in \mathbb{Z}_{12}$ for which $x^2 = 1$. (No proof necesary, but show your work.)
   (c) Find all $x \in \mathbb{Z}_5$ for which $x^2 = 1$. (No proof necesary, but show your work.)
   (d) Prove that if p is a prime number, then 1 and $p - 1$ both are solutions to the equation $x^2 = 1$ in $\mathbb{Z}_p$, and there are *no* other solutions.

6. Recall that a nonzero element a in a ring R is a *zero divisor* if there exists a nonzero element b in R such that $ab = 0$ **or** $ba = 0$. The following problem shows that there is a distinction between *left zero divisors*, *right zero divisors*, and *zero divisors*.

   (a) Consider the set of infinite-dimensional matrices with entries in $\mathbb{R}$ for which each column and each row has only finitely many nonzero entries. Show that this set, which will be denoted by $\mathcal{M}$, is a ring under matrix addition and multiplication. What is the zero element of the ring? Does it have a unity element?

   (b) Define the following 'left shift' matrix, which has 1's above the diagonal and 0's for all other entries:

$$L \quad := \quad \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \\ \vdots & & & & \end{pmatrix}$$

   Define the following 'truncation' matrix, which has a 1 in the top left entry and all other entries are 0:

$$T \quad := \quad \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & & & \end{pmatrix}$$

   Show that the matrix product LT is zero (so L is a left zero divisor in $\mathcal{M}$), but L is not a right zero divisor.

   (c) Define a 'right shift' matrix R, and show that it is a right zero divisor, but not a left zero divisor.

7. Prove that if a matrix $A \in M_n(\mathbb{Q})$ is a left zero divisor (in $M_n(\mathbb{Q})$), then A is also a right zero divisor. (You may use facts from your Linear Algebra class for this problem.)

8. Section 20 #6, 10, 18