Homework 3

Math 415 (section 502), Fall 2015

This homework is due on Thursday, September 17. You may cite results from class, as appropriate. You may also cite facts from Linear Algebra.

0. Re-read the "Some words of warning" subsection of Section 2. Read Sections 3–5.

1. Prove the following *subgroup criterion*: A subset H of a group G is a subgroup if and only if H is nonempty and for all $a, b \in H$, $ab^{-1} \in H$.

2. Recall that $\mathrm{GL}_n(\mathbb{R})$ is the *general linear group*, which consists of all invertible $n \times n$ matrices with entries in $\mathbb{R}$.

   (a) Which binary operation makes $\mathrm{GL}_n(\mathbb{R})$ a group? Which binary operation makes $\mathbb{R}^*$ a group? (No proof necessary.)

   (b) For $n \geq 2$, is $\mathrm{GL}_n(\mathbb{R})$ abelian? Give a proof.

   (c) Let $\mathrm{SL}_n(\mathbb{R})$ be the set of all $n\times$n matrices with entries in $\mathbb{R}$ and determinant 1. Is $\mathrm{SL}_n(\mathbb{R})$ a subgroup of $\mathrm{GL}_n(\mathbb{R})$? Give a proof.

   (d) Is the set of all $n\times$n matrices with entries in $\mathbb{R}$ and determinant *not* equal to 1 a subgroup of $\mathrm{GL}_n(\mathbb{R})$? Give a proof.

   (e) Let n be a positive number with $n \geq 2$. Is the following function ϕ a group homomorphism? Is it an isomorphism? Give a proof. (The group operations are the ones from part (a).)

$$\begin{aligned} \phi \ : \ \mathrm{GL}_n(\mathbb{R}) &\to \mathbb{R}^* \\ A &\mapsto \det A \end{aligned}$$

3. For a positive integer n, define $\star_n$ on $\mathbb{Z}_n$ to be multiplication modulo n. Prove that if $n = 10$, the set $\{1, 3, 7, 9\}$ is a group with binary operation $\star_n$. (Do not forget to check that $\star_n$ is an associative binary operation.) Write the group table.

4. Are $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ isomorphic groups? Give a proof. (*Hint*: Prove that any homomorphism $\phi : \mathbb{Z} \to \mathbb{Q}$ is *not* onto by consider $\frac{1}{2}\phi(1)$.)

5. Section 4 # 6, 20, 25, 28, 32

6. Section 5 # 14, 18