Homework 8

Math 415 (section 502), Fall 2015

This homework is due on Thursday, October 22. You may cite results from class/homework/exam.

0. *(This problem is not to be turned in.)*
   - (a) Read Section 13.
   - (b) Section 13 # 1–28, 45 (do as many as you can; some are to be turned in – see below)
   - (c) Prove that if H is a subgroup of a group G, then the inclusion function $i : H \to G$ given by $i(x) := x$ is a homomorphism. What is the kernel?
   - (d) List all homomorphisms $\mathbb{Z}_{10} \to A_4$, and their kernels and images.
   - (e) List all homomorphisms $\mathbb{Z} \to A_4$ for which the kernel is $\langle 2 \rangle$.
   - (f) If G, H, and K are finite groups of order m, n, and p, respectively, then what is the order of $G \times H \times K$?
   - (g) If G is an infinite group, and H is a finite group, does it follow that $G \times H$ is an infinite group?
   - (h) Prove or disprove: *if $\phi : G \to G'$ is a homomorphism, and G' is abelian, then G is abelian.*
   - (i) Prove or disprove: *if $\phi : G \to G'$ is a homomorphism, and G is abelian, then G' is abelian.*
   - (j) Prove or disprove: *if $\phi : G \to G'$ is a surjective homomorphism, and G' is abelian, then the kernel of ϕ is abelian.*
   - (k) Prove or disprove: *if $\phi : G \to G'$ is a homomorphism, and G' is cyclic, then G is cyclic.*

1. Let G be a group, and let $g \in G$. Consider the function $\phi : G \to G$ given by $\phi(x) = gxg^{-1}$.
   - (a) Prove that ϕ is a homomorphism.
   - (b) Determine the kernel of ϕ.
   - (c) Is ϕ an automorphism? Give a proof. (Recall that an *automorphism* of a group K is an isomorphism from K to K.)

2. Let G and K be groups. Let $\pi : G \times K \to G$ be the *projection* function $\pi(g, k) := g$.
   - (a) Prove that π is a homomorphism.
   - (b) Prove that kernel of π is isomorphic to K.

3. (a) Prove or disprove: *if* $\phi : G \to G'$ *is a homomorphism, and* G' *is infinite, then* G *is infinite.*

   (b) Prove or disprove: *if* $\phi : G \to G'$ *is a homomorphism,* G *is infinite, and* G' *is finite, then* $\ker(\phi)$ *is infinite.*

4. (a) Explain how the symmetric group S_n can be viewed as a subgroup of S_m for any $m \geq n$.

   (b) Are (17) and (1237) in the same left coset of (the subgroup) S_6 in the group S_7? Explain. (*Hint*: Recall the criterion for when 2 cosets are equal.)

   (c) Are (27) and (1237) in the same left coset of (the subgroup) S_6 in the group S_7? Explain.

5. Section 13 # 10, 22, 29, 32, 40, 50, 52

6. Section 14 # 6

7. (Challenge problem – optional!)

   (a) Prove or disprove: *if* G *is an abelian group that is not cyclic, then* G *contains a subgroup isomorphic to* $\mathbb{Z}_p \times \mathbb{Z}_p$ *for some prime number* p. (Last week's homework did the case when G is finite.)

   (b) Prove or disprove: *for a subgroup* H *of a group* G*, every left coset of* H *contains the identity element of* G*.*

   (c) Prove or disprove: *for a subgroup* H *of a group* G*, if two left cosets of* H *intersect, then they are equal.*

   (d) Section 13 # 53