Homework 9

Math 415 (section 502), Fall 2015

This homework is due on Thursday, October 29. You may cite results from class/homework/quiz/exam.

0. *(This problem is not to be turned in.)*

   (a) Read Section 14.

   (b) Section 14 # 6, 22, 35, 38, 40

1. True/False. (No proofs necessary for this problem.)

   (a) $\mathbb{Z}_3 \times \mathbb{Z}_{10}$ is cyclic.

   (b) $\mathbb{Z} \times \mathbb{Z}_5$ is cyclic.

   (c) $\mathbb{Z}_6 \times \mathbb{Z}_{21}$ is cyclic.

   (d) $\mathbb{Z}_6 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_{12}$.

   (e) Every subgroup of $\mathbb{Z} \times \mathbb{Z}_5 \times \mathbb{R}$ is normal.

   (f) If $\phi : G \to G'$ is a surjective homomorphism, then $G/\ker(\phi) \cong G'$.

   (g) Assume $H \trianglelefteq G$ (i.e., H is a normal subgroup of G). If H is abelian and G/H is abelian, then G is abelian.

   (h) Assume $H \trianglelefteq G$. If H is finite and G/H is finite, then G is finite.

   (i) For any group G, the factor group G/G is isomorphic to $\mathbb{Z}_1$.

   (j) For any group G, the factor group $G/\{e\}$ is isomorphic to G.

2. (No proofs necessary for this problem.)

   (a) What is the order of $(1, 1)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$?

   (b) Why is $\langle(1, 1)\rangle$ a normal subgroup of $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$?

   (c) Explain briefly why $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle(1, 1)\rangle$ is isomorphic to a direct product of cyclic groups of prime-power order.

   (d) What is the order of $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle(1, 1)\rangle$?

   (e) Write down an isomorphism between $\mathbb{Z}_{10} \times \mathbb{Z}_{15}/\langle(1, 1)\rangle$ and a product as described in part (b). Show your work.

3. Assume $H \trianglelefteq G$. Prove that if G is finite, then G/H is finite. (*Hint*: Prove that $|G/H| = \frac{|G|}{|H|}$.)

4. Let G be the group of all functions $\mathbb{R} \to \mathbb{R}$. In class, we saw that $H := \{f \in G \mid f(5) = 0\}$ is a normal subgroup of G. Use the fundamental homomorphism theorem to prove that $G/H \cong \mathbb{R}$.

5. Let G be a group. Recall that $\mathrm{Aut}(G)$, the set of all automorphisms of G, forms a group under composition. Recall from a previous homework that for all $g \in G$, the function $i_g : G \to G$ given by $i_g(x) := gxg^{-1}$ is an automorphism of G. Let $I_G := \{i_g \mid g \in G\}$.

   (a) Prove that $I_G \leq \mathrm{Aut}(G)$.

   (b) Prove that $I_G \trianglelefteq \mathrm{Aut}(G)$.

6. Assume that H and K are both normal subgroups of a group G and that $K \subset H$. Prove that K is a subgroup of H and, further, that K is a *normal* subgroup of H.

7. Section 14 # 14, 23

8. (Challenge problem – optional!)

   (a) Section 14 #39

   (b) Let H be a subgroup of G. Prove that if the number of left cosets of H in G is 2, then $H \trianglelefteq G$

   (c) Let G be the group of permutations of $\mathbb{Z}$. Let

$$H \ := \ \{\phi \in G \mid \phi(a) = a \text{ for all } a \leq 0\} \ .$$

   Let $\sigma \in G$ be the permutation defined by $\sigma(a) := a + 1$. Prove the following:

   (1) $H \leq G$,

   (2) $\sigma h \sigma^{-1} \in H$ for all $h \in H$, and then conclude from the exam that $\sigma H \subseteq H\sigma$,

   (3) $\sigma H \subsetneq H\sigma$, and then conclude that H is *not* a normal subgroup of G.