Homework 14 (the last one)

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, April 27. *Write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)* Read Section 54

1. Suggest a problem for the final exam that pertains to Galois Theory.

2. True/False. (You do *not* need to give proofs for this problem.)

   (a) If $\alpha^2 = \beta^2$, then $F(\alpha) = F(\beta)$.

   (b) The splitting field for $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, i)$.

   (c) $\mathbb{F}_{p^n} \cap \mathbb{F}_{p^m} = \mathbb{F}_{p^{\gcd(n,m)}}$.

   (d) For any extension K/F, every automorphism of K restricts to an automorphism of F.

3. Complete the following:
   Claim: Let F be a finite field of order p^r (where p is a prime number). Let K be a degree-n extension of F (where n is a positive integer). Then K is normal over F, and $G(K/F)$ is the cyclic group of order ______ generated by ____________.

4. Prove that a finite normal extension K over F has *no* intermediate fields if and only if the degree $[K : F]$ is a prime number.

5. Let $F := \mathbb{Q}(i)$, and let $E := \mathbb{Q}(i, \sqrt[4]{2})$.

   (a) What is the index $\{E : F\}$? Explain.

   (b) What is a basis of E as an vector space over F? (You do *not* need to give a proof.)

   (c) Is E a normal extension of F? Explain.

   (d) Determine the group $G(E/F)$, and draw the diagram of all intermediate fields L with $F \subset L \subset E$. Give a proof.

6. Let K be the splitting field of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.

   (a) Compute $[K : \mathbb{Q}]$, and state a basis of K as a vector space over $\mathbb{Q}$.

   (b) Is K a normal extension of $\mathbb{Q}$? Explain.

   (c) Which group is $G(K/\mathbb{Q})$ isomorphic to? Give a generating set of $G(K/\mathbb{Q})$. Explain.

   (d) Is there a subgroup H of $G(K/\mathbb{Q})$ for which the fixed field K_H equals $\mathbb{Q}(\sqrt{15})$? If yes, find it. If no, explain why not.

7. Assume that K is a normal extension of F, and $G(K/F)$ is isomorphic to S_4. How many fields E are there for which $F \subset E \subset K$ and $[K : E] = 2$? (You may look up and use any information about S_4.)

8. Assume that $F \subset K$ is a field extension where $|F| = 3^2$ and $|K| = 3^{40}$. Give the subfield diagram for the subfields E where $F \subset E \subset K$.

9. (Honors only!) Assume that F is a finite field. Assume that $F \subset E \subset \overline{F}$ is a field extension. Prove that E is a splitting field over F. (Do *not* assume that E is finite over F.)

10. (Honors only!) *Prove or disprove*: For any subfield K of $\mathbb{C}$, the complex-conjugation automorphism of $\mathbb{C}$ restricts to an automorphism of K.