Homework 4

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, February 9. *As a reminder, please write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Sections 29 and 30.
   (b) (Practice Problems) Section 29 #10, 24, 29
   (c) (Practice Problems) Section 30 #6, 26
   (d) For each of the following, give an example of such an ideal (with explanation) or prove that no such example exists:
      (i) a prime ideal in $\mathbb{Z}_5[x]$ that is not a principal ideal.
      (ii) a principal ideal in $\mathbb{Z}_5[x]$ that is not a maximal ideal.
      (iii) a maximal ideal in $\mathbb{Z}_5[x]$ that is not a prime ideal.
      (iv) a prime ideal in $\mathbb{Z}$ that is not a maximal ideal.
1. Suggest a problem for the Midterm Exam (pertaining to Sections 26–27, 29–30).
2. (No proofs necessary for this problem)
   (a) List all ideals of $\mathbb{Z}$ that contain the number 18.
   (b) List all **maximal** ideals of $\mathbb{Z}$ that contain the number 18.
   (c) List all ideals of $\mathbb{Q}[x]$ that contain $x^2(x^2+1)$.
   (d) List all **prime** ideals of $\mathbb{R}[x]$ that contain $x^2(x^2+1)$.
   (e) List all **maximal** ideals of $\mathbb{C}[x]$ that contain $x^2(x^2+1)$.
3. (a) Which concept from Section 29 do you find the most difficult to understand? Explain your answer briefly.
   (b) List all results, definitions, and examples – excluding Theorem 30.23 – in Section 30 that you did *not* cover in a previous course. (If there are none, write "none".)
4. Complete the following sentence:
   Viewed as a vector space over the field ____________, *the ring* $\mathbb{Z}_2[x]/\langle x^3+x+1\rangle$ *has as a basis the set* {__________________________} *and hence the number of elements in the ring is* ____________.
5. Section 29 # 6, 23
6. Section 30 # 15

7. (a) Any extension field E of a field F can be viewed as an F-vector space via what scalar multiplication? (You do *not* need to give a proof.)

   (b) Let E be a field extension of a field F such that E is an n-dimensional F-vector space (where n is a positive integer). Prove that every element $\alpha \in E$ is algebraic over F. (*Hint:* Consider the elements $1, \alpha, \alpha^2, \ldots$.)

   (c) Prove that if $\alpha \in E$ is transcendental over F, then $F(\alpha)$ is an infinite-dimensional vector space over F. (*Hint:* Use (b).)

8. Let E be an extension field of a field F, and let $\alpha \in E$. Prove that α is algebraic over F if and only if α^2 is algebraic over F.

9. Prove or disprove the following:
   Claim: $\mathbb{Q}(i) = \mathbb{C}$.

10. (Regular-section only!) Consider the ideal $I = \langle x - 3 \rangle$ in $\mathbb{C}[x]$. Does $x + I$ have a multiplicative inverse in the factor ring $\mathbb{C}[x]/I$? If so, compute it (and prove your answer). If not, explain why not.

11. (Honors only!) Let F be a field, and assume that $p = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$ is a non-constant *irreducible* polynomial over F such that $a_0 \neq 0$. Does $x + \langle p \rangle$ necessarily have a multiplicative inverse in the factor ring $F[x]/\langle p \rangle$? If so, compute it (and prove your answer). If not, explain why not.

12. (Honors only!) Let F be a field, and assume that $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in F[x]$ is a non-constant irreducible polynomial over F. Let $E := F[x]/\langle p \rangle$, and define

$$\psi : \quad \begin{aligned} F &\to E \\ a &\mapsto a + \langle p \rangle \end{aligned}$$

   (a) Explain why $\psi[F]$ is a field, and then conclude that E is a field extension of $\psi[F]$.

   (b) For $\alpha := x + \langle p \rangle \in E$, what is $\text{irr}(\alpha, \psi[F])$? Give a proof.