Homework 7

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, March 2. *Write your section number on top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Section 16
   (b) (Practice Problems) Section 16 #7, 13
1. Section 16 #8, 12
2. Determine whether the following operations define group actions. Prove your answers.
   (a) $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ given by $(a, x) \mapsto x - a$
   (b) $\mathbb{Z}_9 \times \mathbb{R} \to \mathbb{R}$ given by $(\bar{a}, x) \mapsto x - a$, where $\mathbb{Z}_9 = \{\bar{0}, \bar{1}, \ldots, \bar{8}\}$.
   (c) $\mathbb{Q}[x] \times \mathbb{Q} \to \mathbb{Q}$ given by $(f, a) \mapsto f(a)$, where we recall that $\mathbb{Q}[x]$ is a group under addition.
3. Let $M_n(\mathbb{Q})$ denote the $n \times n$-matrices with entries in $\mathbb{Q}$. Let $GL_n(\mathbb{Q})$ denote the *invertible* $n \times n$-matrices with entries in $\mathbb{Q}$.
   (a) Which operation makes $GL_n(\mathbb{Q})$ into a group? (No proof necessary.)
   (b) Prove that conjugation defines a $GL_n(\mathbb{Q})$-action on $M_n(\mathbb{Q})$.
   (c) Let $B \in M_n(\mathbb{Q})$. Prove that B is diagonalizable over $\mathbb{Q}$ if and only if *every* matrix C in the orbit of B is diagonalizable over $\mathbb{Q}$.
4. Consider the ring $R = \mathbb{Z}_2[x]/\langle x^3 + x^2 + 1\rangle$.
   (a) Is R a field? What is the order (size) of R? Explain.
   (b) Is R isomorphic to $GF(2^3)$? Explain.
   (c) Which group is $(R^*, \cdot)$ isomorphic to? Explain.
   (d) Which group is $(R, +)$ isomorphic to? Prove your answer.
5. (Honors only!)
   (a) Let $f \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree d. For any integer $n \geq 1$, prove that $f | x^{p^n} - x$ if and only if $d|n$.
   (*Hint*: consider an extension of $\mathbb{Z}_p$ that contains a root of f.)
   (b) Show that for any positive integer n, the polynomial $x^{p^n} - x \in \mathbb{Z}_p[x]$ factors into irreducibles as
   $$x^{p^n} - x \quad = \prod_{d|n} \prod_{\substack{f \in \mathbb{Z}_p[x] \\ f \text{ monic and irreducible in } \mathbb{Z}_p[x] \\ \deg f = d}} f(x)$$
   (c) Let $g \in \mathbb{Z}_p[x]$ be any polynomial of degree n. Prove that g is irreducible in $\mathbb{Z}_p[x]$ if and only if $g | x^{p^n} - x$ and $\gcd(g, x^{p^d} - x) = 1$ for all positive integers d with $d|n$ and $d < n$.