Homework 10

Math 653, Fall 2019

This homework is due on Thursday, October 31.

1. Read Hungerford, Section 2.7
   (a) Section 2.7 #3, 4
   (b) *(These problems are not to be turned in.)* Section 2.5 #11,
   (c) *(These problems are not to be turned in.)* Section 2.7 #1, 2, 6, 9
2. Up to isomorphism, how many groups of order 175 are there? Prove your answer.
3. Let G be a finite group.
   (a) *Prove or disprove:* If H is a Sylow subgroup of G, then $N_G(H) = H$.
   (b) *Prove or disprove:* If H is a Sylow subgroup of G, then $N_G(N_G(H)) = N_G(H)$.
4. Let p be an odd prime.
   (a) Find a generating set for a Sylow p-subgroup of the symmetric group S_{2p}. (Prove your answer.)
   (b) Which (known) group is your answer to (a) isomorphic to? Explain.
5. (a) Let G be a group. Assume that H and K are subgroups of G, with H normal in G, such that $H \cap K = \langle e \rangle$. Prove that $HK \cong H \rtimes K$ (with respect to some group homomorphism $\theta : K \to \mathrm{Aut}(H)$). (*Hint:* Consider conjugation.)
   (b) *Prove or disprove*: If a group of order 80 is *not* nilpotent, then it is a semidirect product. (*Hint:* We partially analyzing groups of order 80 in class.)
6. (a) Which of the symmetric groups S_1, S_2, S_3, and S_4 are solvable? Explain.
   (b) Is S_4 nilpotent? Explain.
7. (a) Are subgroups of nilpotent groups, also nilpotent? Explain.
   (b) Are quotient groups of nilpotent groups, also nilpotent? Explain.
   (c) If a normal subgroup N of a group G is nilpotent, and G/N is nilpotent, does this imply that G is nilpotent? Explain.