Homework 7

Math 653, Fall 2019

This homework is due on Thursday, October 10.

1. Read Hungerford, section 2.1.
   (a) Section 2.1 #5, 7, 8, 9, 11
   (b) *(These problems are not to be turned in.)* Section 2.1 #1, 2, 3
   (c) *(This problem is not to be turned in.)* Prove or disprove: Let $a, b \in \mathbb{Z}$ with $a \neq b$. If $\{x_1, \ldots, x_n\}$ is a basis of a free abelian group F, then so is $\{x_1 + ax_2,\ x_2 + bx_1,\ x_3, \ldots, x_n\}$.
2. (a) Is the group $\mathbb{Q}^2$ isomorphic to $\mathbb{Q}$? Prove your answer.
   (b) Is $\mathbb{Q}$ a free abelian group? Prove your answer.
3. Give an example of the following (and prove your answers):
   (a) A *linearly independent* subset of $\mathbb{Z}^3$ that can NOT be extended to a basis of $\mathbb{Z}^3$.
   (b) A *generating set* of $\mathbb{Z}^3$ that does NOT contain a basis of $\mathbb{Z}^3$.
4. *Prove or disprove:* If a free abelian group G is generated by n elements, then the rank of G is at most n.
5. *Prove the part of the proof of Theorem 1.6 that we skipped in class*: Let F be a free abelian group with basis $\{x_1,\ y_2,\ ,\ldots, y_n\}$. Let G be a nontrivial subgroup of F. Assume $v := d_1x_1 \in G$, where d_1 is the minimal element of the set S of all positive integers s for which there exists a basis $\{z_1, \ldots, z_n\}$ of F and integers $k_i \in \mathbb{Z}$ such that $sz_1 + (k_2z_2 + \cdots + k_nz_n) \in G$. Let H be the free abelian subgroup of F generated by $\{y_2, \ldots, y_n\}$. Then:
$$\langle v \rangle + (G \cap H) \ = \ G\ .$$
6. Let $G := \{(4m + 10n,\ 6m + 20n) \mid m, n \in \mathbb{Z}\}$. Show that G is a subgroup of $F := \mathbb{Z}^2$, and then find a basis $\{x_1, x_2\}$ of F and positive integers d_1 and d_2 satisfying the conditions of Theorem 1.6 (for this F and G). Prove your answer.
7. *(Challenge problem: optional).* Consider $G := 5\mathbb{Z} \times 6\mathbb{Z} < F := \mathbb{Z} \times \mathbb{Z}$. For this subgroup/group, find a basis $\{x_1, x_2\}$ of F and positive integers d_1 and d_2 satisfying the conditions of Theorem 1.6.