Homework 9

Math 653, Fall 2019

This homework is due on Thursday, October 24.

1. Read Hungerford, sections 2.4 and 2.5. Skim section 2.6
   (a) What is an *inner automorphism*?
   (b) Section 2.4 #7, 8, 10
   (c) Section 2.5 #6, 8, 9
   (d) Section 2.6 #1
   (e) *(These problems are not to be turned in.)* Section 2.4 #13
   (f) *(These problems are not to be turned in.)* Section 2.5 #1, 10
2. Does the function $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, given by $(a, x) \mapsto ax$, define a group action? Explain.
3. Let G be a finite group with $|G| = p(q_1 q_2 \dots q_r)$ where p is prime and the q_i's are prime. *Prove or disprove*: If $p > q_1 q_2 \dots q_r$, then every order-p subgroup of G is normal.
4. *Prove or disprove*: Let $G = \langle g \rangle$ be a cyclic group of (finite) order n, acting on a set S. Let $x \in S$. Then there exists a divisor d of n such that (a) the elements $x, gx, \dots, g^{d-1}x$ are distinct, (b) $g^d x = x$, and (c) the orbit of x is $\{x, gx, \dots, g^{d-1}x\}$.
5. Let G be a finite group acting on a finite set S of size at least 2. Assume that G acts *transitively* on S, that is, for every $x, y \in S$, there exists $g \in G$ such that $gx = y$.
   (a) Let $x \in S$. Prove that the orbit of x is S.
   (b) Let $x, y \in S$. Prove that there exists $g \in G$ such that $gG_xg^{-1} = G_y$. (Recall that G_x denotes the stabilizer of x.)
   (c) Let $x \in S$. Prove that $|S| = [G : G_x]$, and conclude that $|S|$ divides $|G|$.
6. Compute the *centralizers* in S_5 of (1234) and of $(123)(45)$. Prove your answers.
7. Prove the alternating version of Cayley's Theorem: *Every finite group is isomorphic to a subgroup of A_n for some n.*
8. Find all Sylow 2-subgroups of S_4. Which (known) group is each isomorphic to?
9. Prove that every order-132 group is *not* simple.
10. Prove that if G is a group of order 3825, then every normal subgroup of order 17 is contained in the center of G.
11. Suggest a problem for the next exam (which is on Thursday, November 7) pertaining to any topic in Chapter 2.