Section 2.3

Compute the exact value of the following limits. If the limit does not exist, support your answer by evaluating left and right hand limits.

1. $\lim_{x\to 1}(4x^3-3x+1) = 4(1)^3 - 3(1) + 1$

$= \boxed{2}$

2. $\lim_{x\to -5}\frac{x^2+5x}{x+5} = \frac{0}{0} \longrightarrow$ Algebra!

$\lim_{x\to -5}\frac{x\cancel{(x+5)}}{\cancel{x+5}} = \lim_{x\to -5}(x)$

$= \boxed{-5}$

3. $\lim_{x\to 2}\frac{x-\sqrt{3x-2}}{x^2-4}\cdot\frac{x+\sqrt{3x-2}}{x+\sqrt{3x-2}} = \lim_{x\to 2}\frac{x^2-(3x-2)}{(x^2-4)(x+\sqrt{3x-2})}$

$\to x^2-3x+2$

$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x-1)}{(x+2)\cancel{(x-2)}(x+\sqrt{3x-2})}$

$= \frac{1}{4(2+2)}$

$= \boxed{\frac{1}{16}}$

4. $\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$

$$\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} \cdot \frac{(3+h)(3)}{(3+h)(3)} = \lim_{h\to 0} \frac{3-(3+h)}{h(3+h)(3)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3+h)(3)}$$

$$= \lim_{h\to 0} \frac{-1}{(3+h)(3)}$$

$$= \boxed{-\frac{1}{9}}$$

5. $\lim_{x\to 1} \frac{x-4}{x-1} = \frac{-3}{0}$ $\frac{\text{no zero}}{\text{zero}}$

$$\lim_{x\to 1^+} \frac{x-4}{x-1} \quad \frac{-}{+} = -\infty$$

$$\lim_{x\to 1^-} \frac{x-4}{x-1} \quad \frac{-}{-} = \infty$$

$$\lim_{x\to 1^+} \frac{x-4}{x-1} \neq \lim_{x\to 1^-} \frac{x-4}{x-1}$$

$\boxed{\text{limit does not exist}}$

6. $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} x+5 & \text{if } x \le 3 \\ x^3 - 3 & \text{if } x > 3 \end{cases}$

- $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3 - 3) = 24$

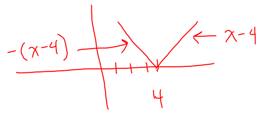
  $x > 3$

- $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+5) = 8$

$$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$$

$\boxed{\text{limit does not exist}}$

warm up : $\displaystyle\lim_{x\to 4} \frac{x-4}{|x-4|}$



graph of $|x-4|$

$$|x-4| = \begin{cases} x-4 & x \geq 4 \\ -(x-4) & x<4 \end{cases}$$

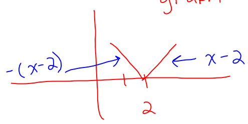
- $\displaystyle\lim_{x\to 4^+} \frac{x-4}{|x-4|} = \lim_{x\to 4^+} \frac{x-4}{x-4} = 1$

- $\displaystyle\lim_{x\to 4^-} \frac{x-4}{|x-4|} = \lim_{x\to 4^-} \frac{x-4}{-(x-4)} = -1$

$\boxed{\text{thus } \displaystyle\lim_{x\to 4} \frac{x-4}{|x-4|} \text{ dne}}$

8. $\displaystyle\lim_{x\to 2} \frac{x^2-4}{|x-2|}$

graph of $|x-2|$



$\displaystyle\lim_{x\to 2^+} \frac{(x-2)(x+2)}{|x-2|}$

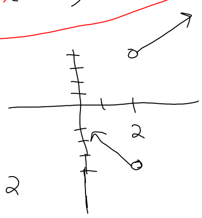
- $\displaystyle\lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$

$\boxed{\therefore \displaystyle\lim_{x\to 2} \frac{x^2-4}{|x-2|} \text{ dne}}$

- $\displaystyle\lim_{x\to 2^-} \frac{(x-2)(x+2)}{|x-2|} = \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x+2)}{-\cancel{(x-2)}} = -4$

Graph of $\dfrac{(x-2)(x+2)}{|x-2|}$

$$= \begin{cases} x+2 & x>2 \\ -(x+2) & x<2 \end{cases}$$



7. $\displaystyle\lim_{x\to 0^-} \left(\underset{-\infty}{\frac{1}{x}} - \underset{\infty}{\frac{1}{|x|}}\right) = -\infty - \infty = \boxed{-\infty}$

Aside : $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{x}\right) = -\infty$

$\displaystyle\lim_{x\to 0^-} \frac{1}{|x|} = \infty$

if $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing a

and $\displaystyle\lim_{x\to a} f(x) = L$ Then $\displaystyle\lim_{x\to a} g(x) = L$

$\displaystyle\lim_{x\to a} h(x) = L$

9. $\displaystyle\lim_{x\to 1} f(x)$ if it is known that $\underset{4}{4x} \leq f(x) \leq \underset{4}{x+3}$ for all x in $[0, 2]$.

Squeeze theorem : $\displaystyle\lim_{x\to 1} (4x) = 4$, $\displaystyle\lim_{x\to 1} (x+3) = 4$ } same!

$\therefore \displaystyle\lim_{x\to 1} f(x) = 4$

Section 2.5

10. Referring to the graph, explain why the function $f(x)$ is or is not continuous (you decide which) at $x = -1$, $x = 3$, $x = 5$, $x = -4$ and $x = 7$. For the values of x where $f(x)$ is not continuous, is it continuous from the right, left or neither? In addition, for each discontinuity, is it a jump discontinuity, infinite discontinuity or a removable discontinuity?

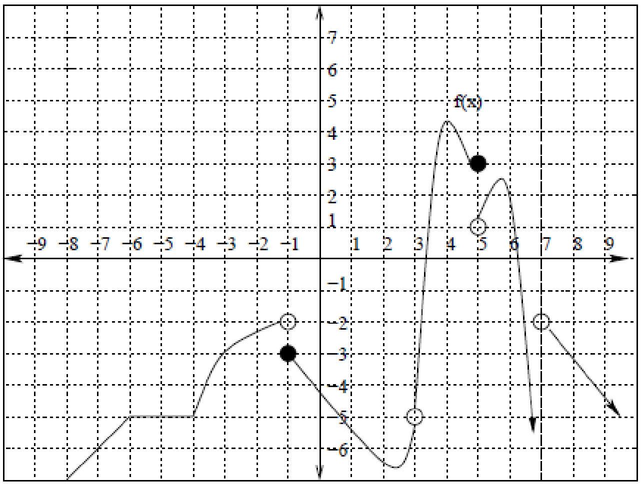
$f(x)$ is continuous at $x = a$ if all of the following is true

① $f(a)$ must exist

② $\lim_{x \to a} f(x)$ must exist

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

$\boxed{③ \ \lim_{x \to a} f(x) = f(a)}$



at $x = -1$: $\lim_{x \to -1^+} f(x) = -3$

$\lim_{x \to -1^-} f(x) = -2$

not equal

$\lim_{x \to -1} f(x)$ dne, thus not continuous at $x = -1$

at $x = 3$: $f(3)$ is not defined thus not continuous at $x = 3$.

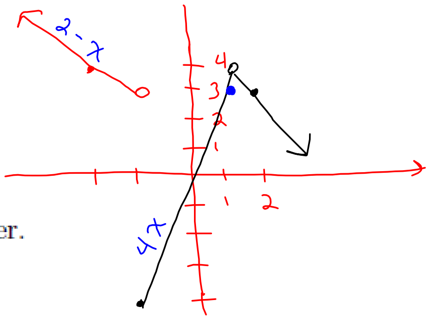
at $x = 5$: $\lim_{x \to 5^+} f(x) = 1$, $\lim_{x \to 5^-} f(x) = 3$, so $\lim_{x \to 5} f(x)$ dne

at $x = 7$: $f(7)$ dne

11. Sketch the graph of $f(x)$ and determine where the function

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ 4x & -1 \leq x < 1 \\ 3 & x = 1 \\ 5 - x & \text{if } x > 1 \end{cases}$$

is not continuous. Fully support your answer.



not continuous at $x = -1$
because $\lim_{x \to -1} f(x)$ dne

not continuous at $x = 1$ because $\lim_{x \to 1} f(x) = 4$
but $f(1) = 3$

12. Which of the following functions has removable discontinuity at $x = a$? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at $x = a$. Note: f has removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists and $f(x)$ can be redefined so that $\lim_{x \to a} f(x) = f(a)$ (thereby removing the discontinuity).

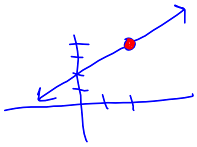
(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x = 2$.

$f(2)$ does not exist because $x=2$ is not in the domain.

not continuous at $x=2$.

is the continuity removable?

① Does $\lim_{x \to 2} \dfrac{x^2-4}{x-2}$ exist?



$$\lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = 4$$

remove discontinuity

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

$\boxed{f(x) = x+2}$

(b) $f(x) = \dfrac{1}{x - 1}$, $x = 1$

not continuous at $x=1$ because $f(1)$ dne.

$\lim_{x \to 1} \dfrac{1}{x-1}$ dne because $\lim_{x \to 1^+} \dfrac{1}{x-1} = \infty$

not removable!

$\lim_{x \to 1^-} \dfrac{1}{x-1} = -\infty$

(c) $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x + 4 & \text{if } x \geq 1 \end{cases}$, $x = 1$

$\lim_{x \to 1^+} f(x) = 6$

$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1} f(x)$ dne

continuity cannot be removed.

13. If $f(x) = \dfrac{x+2}{x^2+5x+6}$, find all values of $x = a$ where the function is discontinuous. For each discontinuity, find the limit as x approaches a, if the limit exists. If the limit does not exist, support your answer by evaluating left and right hand limits.

$$f(x) = \frac{x+2}{(x+2)(x+3)}$$

not continuous at $x=-2$ + $x=-3$

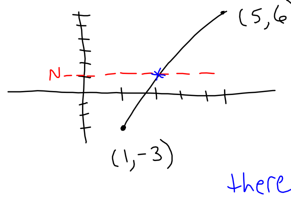
at $x=-2$: $\displaystyle\lim_{x\to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x+3)} = 1$

at $x=-3$: $\displaystyle\lim_{x\to -3} \frac{x+2}{(x+2)(x+3)} = \lim_{x\to -3} \frac{1}{x+3}$ $\boxed{\text{dne}}$

$$\lim_{x\to -3^+} \frac{1}{x+3} = \infty$$

$$\lim_{x\to -3^-} \frac{1}{x+3} = -\infty$$

14. Suppose it is known that $f(x)$ is a continuous function defined on the interval $[1,5]$. Suppose further it is given that $f(1) = -3$ and $f(5) = 6$. Give a graphical arguement that there is at least one solution to the equation $f(x) = 1$. ← $N=1$

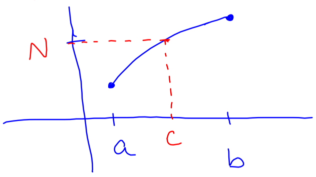


$f(5) > 1$

$f(1) < 1$

there is a solution to $f(x)=1$ on $[1,5]$

Intermediate value theorem: If $f(x)$ is continuous on $[a,b]$ if N is any number between $f(a)$ and $f(b)$, Then there exist a value of c on $[a,b]$ so that $f(c)=N$



15. If $g(x) = x^5 - 2x^3 + x^2 + 2$, use the Intermediate Value Theorem to find an interval which contains a root of $g(x)$, that is contains a solution to the equation $g(x) = 0$.

$$g(x) = x^5 - 2x^3 + x^2 + 2$$

To show there is a solution to $g(x)=0$ "get on either side of $y=0$"

$\boxed{g(0) = 2 > 0}$

$g(-1) = -1+2+1+2 = 4 > 0$:(

$\boxed{g(-2) = -32+16+4+2 = -10 < 0}$

$g(0) > 0$ $g(x)=0$ exists on $[-2,0]$

$g(-2) < 0$

16. Find the values of c and d that will make

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ cx^2 + d & \text{if } 1 \le x \le 2 \leftarrow \\ 4x & \text{if } x > 2 \end{cases}$$

continuous on all real numbers. Once the value of c and d is found, find $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} f(x)$.

$\lim_{x \to 1^+} f(x) = c + d$

$\lim_{x \to 1^-} f(x) = 2$

$\lim_{x \to 2^+} f(x) = 8$

$\lim f(x) = 4c + 2$

$$\boxed{\begin{matrix} c + d = 2 \\ \\ 8 = 4c + d \end{matrix}}$$

$\rightarrow \underbrace{d = 2 - c}$

$\rightarrow 8 = 4c + 2 - c$

$6 = 3c \rightarrow c = 2$

$d = 0$

16. Find the values of c and d that will make

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ cx^2 + d & \text{if } 1 \le x \le 2 \\ 4x & \text{if } x > 2 \end{cases} \longrightarrow f(x) = \begin{cases} 2x & x < 1 \\ 2x^2 & 1 \le x \le 2 \\ 4x & x > 2 \end{cases}$$

continuous on all real numbers. Once the value of c and d is found, find $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} f(x)$.

$\lim_{x \to 1} f(x) = 2$; $\lim_{x \to 2} f(x) = 8$

Section 2.6

17. Compute the following limits:

a.) $\lim_{x\to\infty} \frac{4x^3 - 6x^4}{2x^3 - 9x + 1} = \lim_{x\to\infty} \frac{\cancel{x^4}^{\,x}\left(\frac{4}{x} - 6\right)}{\cancel{x^3}\left(2 - \frac{9}{x^2} + \frac{1}{x^3}\right)}$

Factor out the higher power of x on the top & bottom

$= \lim_{x\to\infty} \frac{x\left(\frac{4}{x}^{\to 0} - 6\right)}{2 - \frac{9}{x^2}^{\to 0} + \frac{1}{x^3}^{\to 0}} = \frac{(\infty)(-6)}{2} = \boxed{-\infty}$

b.) $\lim_{t\to-\infty} \frac{t^9 - 4t^{10}}{t^{12} + 2t^2 + 1}$

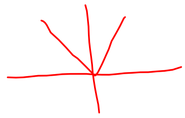
$\lim_{t\to-\infty} \frac{\cancel{t^{10}}\left(\frac{1}{t} - 4\right)}{t^2\,\cancel{t^{12}}\left(1 + \frac{2}{t^{10}} + \frac{1}{t^{12}}\right)} = \lim_{t\to-\infty} \frac{\frac{1}{t}^{\to 0} - 4}{t^2\left(1 + \frac{2}{t^{10}} + \frac{1}{t^{12}}\right)}$

$= \frac{-4}{\infty} = \boxed{0}$

c.) $\lim_{x\to\infty} \frac{4x - 6x^3}{-2x^3 - 9x + 1} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{4}{x^2}^{\to 0} - 6\right)}{\cancel{x^3}\left(-2 - \frac{9}{x^2}^{\to 0} + \frac{1}{x^3}^{\to 0}\right)} = \frac{-6}{-2} = \boxed{3}$

d.) $\lim_{x\to\infty} \frac{\sqrt{2+x^2}}{4-7x}$

note: $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



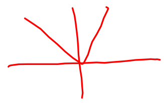
$$\lim_{x\to\infty} \frac{\sqrt{x^2\left(\frac{2}{x^2}+1\right)}}{x\left(\frac{4}{x}-7\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{\frac{2}{x^2}+1}}{x\left(\frac{4}{x}-7\right)}$$

here, $|x| = x$ since $x \to +\infty$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{\frac{2}{x^2}\to 0+1}}{\cancel{x}\left(\frac{4}{x}-7\right)} = \boxed{\frac{1}{-7}}$$

$\frac{4}{x} \to 0$

e.) $\lim_{x\to-\infty} \frac{\sqrt{5x^2+1}}{x-3}$



$$\lim_{x\to-\infty} \frac{\sqrt{x^2\left(5+\frac{1}{x^2}\right)}}{x\left(1-\frac{3}{x}\right)} = \lim_{x\to-\infty} \frac{|x|\sqrt{5+\frac{1}{x^2}}}{x\left(1-\frac{3}{x}\right)}$$

here, $|x| = -x$ since $x \to -\infty$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{5+\frac{1}{x^2}}}{\cancel{x}\left(1-\frac{3}{x}\right)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{5+\frac{1}{x^2}\to 0}}{1-\frac{3}{x}\to 0} = \boxed{-\sqrt{5}}$$

f.) $\lim_{x\to\infty}(\sqrt{x^2+5x+1}-x)\dfrac{\sqrt{(x^2+5x+1)}+x}{\sqrt{x^2+5x+1}+x} = \lim_{x\to\infty}\dfrac{x^2+5x+1-x^2}{\sqrt{x^2+5x+1}+x}$

conjugate!

$$= \lim_{x\to\infty}\frac{5x+1}{\sqrt{x^2+5x+1}+x} = \lim_{x\to\infty}\frac{x\left(5+\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{5}{x}+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x\to\infty}\frac{x\left(5+\frac{1}{x}\right)}{|x|\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}+x}$$

here, $|x| = x$ since $x \to +\infty$

$$= \lim_{x\to\infty}\frac{x\left(5+\frac{1}{x}\right)}{x\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}+x} = \lim_{x\to\infty}\frac{\cancel{x}\left(5+\cancel{\frac{1}{x}}^{\,0}\right)}{\cancel{x}\left(\sqrt{1+\frac{5}{x}+\frac{1}{x^2}}+1\right)}$$

$\frac{5}{x} \to 0$, $\frac{1}{x^2} \to 0$

$$= \boxed{5}$$

g.) $\lim_{x\to-\infty}(x+\sqrt{x^2+x+2})\dfrac{x-\sqrt{x^2+x+2}}{x-\sqrt{x^2+x+2}}$

$$\lim_{x\to-\infty}\frac{x^2-(x^2+x+2)}{x-\sqrt{x^2+x+2}} = \lim_{x\to-\infty}\frac{-x-2}{x-\sqrt{x^2+x+2}}$$

$$= \lim_{x\to-\infty}\frac{x\left(-1-\frac{2}{x}\right)}{x-\sqrt{x^2\left(1+\frac{1}{x}+\frac{2}{x^2}\right)}}$$

here, $|x| = -x$ since $x \to -\infty$

$$= \lim_{x\to-\infty}\frac{x\left(-1-\frac{2}{x}\right)}{x-\underbrace{|x|}_{-x}\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}}$$

$$= \lim_{x\to-\infty}\frac{x\left(-1-\frac{2}{x}\right)}{x+x\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}}$$

$$= \lim_{x\to-\infty}\frac{\cancel{x}\left(-1-\cancel{\frac{2}{x}}^{\,0}\right)}{\cancel{x}\left[1+\sqrt{1+\cancel{\frac{1}{x}}^{\,0}+\cancel{\frac{2}{x^2}}^{\,0}}\right]}$$

$$= \boxed{\frac{-1}{2}}$$

18. Find all horizontal and vertical asymptotes of $f(x) = \dfrac{x^3}{x^3-x}$

$f(x) = \dfrac{x^3}{x(x^2-1)}$ VA: $x = \pm 1$

HA: $y = 1$