1. Using differentials or a linear approximation, approximate $\sqrt{11}$.

a) $\frac{8}{3}$ b) $\frac{11}{3}$ c) $\frac{23}{6}$ d) $\frac{21}{6}$ e) $\frac{10}{3}$

$f(x) = \sqrt{x}$

$a = 9$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(9) + f'(9)(x-9)$

$f(x) = \sqrt{x} \rightarrow f(9) = 3$

$f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(9) = \frac{1}{6}$

$L(x) = 3 + \frac{1}{6}(x-9)$

$\sqrt{11} \approx L(11)$

$\sqrt{11} \approx 3 + \frac{1}{6}(2)$

$= 3 + \frac{1}{3} = \frac{10}{3}$

$(\cos(x^2))^2$

2. If $f(x) = 3x\cos^2(x^2)$, find $f'(0)$.

a) 0 b) -3 c) 3 d) 1 e) -9

Product rule: $f'(x) = 3\cdot\cos^2(x^2) + 3x\cdot 2(\cos(x^2))(-\sin(x^2)\cdot 2x)$

$f'(0) = 3\cdot 1 + 0 = \boxed{3}$

3. A particle is moving according to the equation of motion $f(t) = t^4 - 4t + 1$, where t is measured in seconds and $f(t)$ is measured in feet. What is the acceleration of the particle at the instant when the particle is at rest?

a) $0\frac{ft}{s^2}$ b) $0\frac{ft}{s}$ c) $12\frac{ft}{s^2}$ d) $12\frac{ft}{s}$ e) $-12\frac{ft}{s^2}$

at rest $\longrightarrow v(t) = 0$

$4t^3 - 4 = 0$

$4t^3 = 4$

$t^3 = 1$

$t = 1$

$a(t) = v'(t)$

$= 12t^2$

$a(1) = 12\ ft/s^2$

4. Let $f(x) = (1+x^2)^{\frac{3}{2}}$. Then $f''(0) =$

a) 3 b) 0 c) 6 d) $\frac{3}{4\sqrt{2}}$ e) $\frac{3}{4}$

$f'(x) = \frac{3}{2}(1+x^2)^{\frac{1}{2}}(2x)$

$f'(x) = 3x(1+x^2)^{\frac{1}{2}}$

$f''(x) = 3(1+x^2)^{\frac{1}{2}} + 3x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$

$f''(0) = 3(1)^{\frac{1}{2}} = \boxed{3}$

5. The function $f(x) = x^3 + 5x - 1$ is one-to-one. Let $g = f^{-1}$. Then $g'(5) =$

a) 8 b) $\frac{1}{80}$ c) $\frac{8}{25}$ d) $\frac{1}{8}$ e) 80

$g'(a) = \frac{1}{f'(g(a))}$

$g'(5) = \frac{1}{f'(g(5))}$

$g(5) = ?$

solve $x^3 + 5x - 1 = 5$

$\boxed{x=1}$ thus $g(5) = 1$

$g'(5) = \frac{1}{f'(1)}$

$f'(x) = 3x^2 + 5$

$= \boxed{\frac{1}{8}}$

6. Given the curve parametrized by $x = t^3 - 3t^2 - 9t + 1$, $y = t^3 + 3t^2 - 9t + 1$, at which point does the curve have a vertical tangent?

a) $(1, -3)$ b) $(6, 12)$ c) $(-10, 6)$

d) $(-1, 3)$ e) $(1, 1)$

looking for vertical or horizontal tangent:

① simplify $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{3t^2 + 6t - 9}{3t^2 - 6t - 9} = \frac{3(t^2 + 2t - 3)}{3(t^2 - 2t - 3)}$

$= \frac{(t+3)(t-1)}{(t-3)(t+1)}$

② vertical tangents

solve $\frac{dx}{dt} = 0$.

$\boxed{t=-1, \ t=3}$ now find the points.

6. Given the curve parametrized by $x = t^3 - 3t^2 - 9t + 1$, $y = t^3 + 3t^2 - 9t + 1$, at which point does the curve have a vertical tangent?

a) $(1, -3)$ b) $(6, 12)$ c) $(-10, 6)$

d) $(-1, 3)$ e) $(1, 1)$

$t = -1$: $x = -1 - 3 + 9 + 1 = 6$, $y = -1 + 3 + 9 + 1 = 12$

Points: $(6, 12)$

$t = 3$: $x = 27 - 27 - 27 + 1 = -26$, $y = 27 + 27 - 27 + 1 = 28$

Also $(-26, 28)$, just not there.

horizontal tangents

solve $\frac{dy}{dt} = 0$.

7. $\lim_{x\to 0} \frac{4\cos x - 4 + 3\sin x}{5x} =$

a) $\frac{4}{5}$ b) $-\frac{4}{5}$ c) $\frac{3}{5}$ d) 1 e) 0

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$

$\lim_{x\to 0}\left(\frac{4\cos x - 4}{5x} + \frac{3\sin x}{5x}\right)$

$\lim_{x\to 0}\left(\frac{4(\cos x - 1)}{5x} + \frac{3}{5}\frac{\sin x}{x}\right) = \boxed{\frac{3}{5}}$ (0, 1)

Aside:

$\lim_{x\to 0}\left(\frac{\sin^2 4x}{x^2}\right)$

$\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\cdot\frac{\sin 4x}{4x}\right)\cdot 16 = \boxed{16}$ (1, 1)

8. Find the slope of the line tangent to the curve given by $y^2 + xy = 8$ at the point $(-2, -2)$.

a) -2 b) $-\frac{10}{3}$ c) $-\frac{1}{3}$ d) -3 e) 0

$2y\frac{dy}{dx} + (1)y + x\frac{dy}{dx} = 0$

$x = -2$

$y = -2$

then solve for $\frac{dy}{dx}$

$2(-2)\frac{dy}{dx} + -2 + -2\frac{dy}{dx} = 0$

$-6\frac{dy}{dx} = 2 \rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{3}}$

9. $\lim_{x\to 5^+} e^{x/(5-x)} =$

a) 0 b) ∞ c) $-\infty$ d) 1 e) e

$\lim_{x\to 5^+} e^{\frac{x}{5-x}} = e^{\left(\lim_{x\to 5^+}\frac{x}{5-x}\right)}$

$\lim_{x\to 5^+} \frac{x}{5-x} = -\infty$

5

$= e^{-\infty}$

$= \frac{1}{e^{\infty}}$

$= \boxed{0}$

10. Let $f(x)$ be a differentiable function and let $g(x) = 3x^2 - 1$. Let $H(x) = f(g(x))$, the composit of f and g. If $f(0) = 1$, $f'(0) = -1$, $f(1) = 3$, $f'(1) = 2$, $f(2) = -1$, $f'(2) = 5$, find $H'(1)$.

a) 30 b) 12 c) -6 d) 6 e) 5

$H(x) = f(g(x))$

$H(x) = f(3x^2 - 1)$

$H'(x) = f'(3x^2 - 1)(6x)$

$H'(1) = f'(2)(6)$

$= 5(6) = \boxed{30}$

11. $\lim_{x\to\infty} 3^{1-x} =$

a) 0 b) ∞ c) $-\infty$ d) 1 e) 3

$\lim_{x\to\infty} 3^{1-x} = 3^{\lim_{x\to\infty}(1-x)}$

$= 3^{-\infty} = \frac{1}{3^{\infty}} = 0$

12. Find the domain of vector function $\mathbf{r}(t) = \left\langle \frac{t}{t^2 - 16}, \sqrt{t-2} \right\rangle$. $= \left\langle \frac{t}{(t+4)(t-4)}, \sqrt{t-2} \right\rangle$

$t \neq \pm 4$ $t - 2 \geq 0$, $t \geq 2$

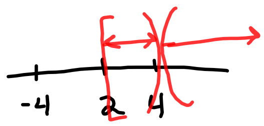
a) $(-\infty, -4) \cup (-4, 2) \cup (2, 4) \cup (4, \infty)$

b) $(-\infty, -4) \cup (4, \infty)$

c) $(-\infty, -4) \cup (-4, 2] \cup [2, 4) \cup (4, \infty)$

d) $[2, 4) \cup (4, \infty)$

e) $(2, 4) \cup (4, \infty)$



$[2, 4) \cup (4, \infty)$

13. If $\mathbf{r}(t) = \langle \cos 3t, t \rangle$ is the position of an object at time t, find the acceleration of the object at time $t = \frac{\pi}{9}$.

a) $\left\langle \frac{1}{2}, 0 \right\rangle$

b) $\left\langle -\frac{1}{2}, 0 \right\rangle$

c) $\left\langle -\frac{9}{2}, 0 \right\rangle$

d) $\left\langle \frac{9}{2}, 0 \right\rangle$

e) $\langle 3, 0 \rangle$

Find $r''(\frac{\pi}{9})$

$r'(t) = \langle -3\sin(3t), 1 \rangle$

$r''(t) = \langle -9\cos(3t), 0 \rangle$

$r''(\frac{\pi}{9}) = \langle -9\cos\frac{\pi}{3}, 0 \rangle$ ($\cos\frac{\pi}{3} = \frac{1}{2}$)

$= \langle -\frac{9}{2}, 0 \rangle$

14. If $f(x) = e^{x \tan x}$, find $f'(x)$.

a) $f'(x) = e^{x \tan x}$

b) $f'(x) = \sec^2 x e^{x \tan x}$

c) $f'(x) = (\tan x + x \sec^2 x)e^{x \tan x}$

d) $f'(x) = (\tan x + x \sec x \tan x)e^{x \tan x}$

e) $f'(x) = x \tan x e^{x \tan x - 1}$

$f'(x) = (x\tan x)' e^{x\tan x}$

$= (\tan x + x\sec^2 x) e^{x\tan x}$

general rule: $\frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)}$

15. Find the equation of the tangent line to the graph of $x = e^{2t}$, $y = te^t$ at the point $(1, 0)$.

a) $y = 2x - 1$

b) $y = 4x - 4$

c) $y = \frac{1}{2}x - \frac{1}{2}$

d) $y = \frac{1}{3}x - \frac{1}{3}$

e) $y = x - 1$

product rule; $(1,0)$ = x, y

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dy}{dx} = \frac{e^t + te^t}{2e^{2t}} \Big|_{t=0}$

$t = 0$: $x = e^{2(0)} = 1$, $y = 0e^0 = 0$

Find t by solving $e^{2t} = 1$, $te^t = 0$

slope : $\frac{dy}{dx} = \frac{1}{2}$

point: $(1, 0)$

equation: $y - 0 = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}x - \frac{1}{2}$

16. Find the quadratic approximation for $f(x) = \frac{1}{x}$ at $x = 1$. $a=1$

a) $x^2 - 3x + 3$

b) $x^2 - x + 2$

c) $x^2 - 2x + 1$

d) $x^2 + 4x + 5$

e) $x^2 + x - 3$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$Q(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= 1 - (x-1) + (x-1)^2$$

$$= 1 - x + 1 + x^2 - 2x + 1$$

$$Q(x) = x^2 - 3x + 3$$

$f = \frac{1}{x}$ $f(1) = 1$

$f' = -\frac{1}{x^2}$ $f'(1) = -1$

$f'' = \frac{2}{x^3}$ $f''(1) = 2$

Part II - Work Out

$(e^t)^2 = e^{2t}$

17. The position of a particle at time t is given by $\mathbf{r}(t) = \left\langle \frac{\cos t}{e^t}, \frac{\sin t}{e^t} \right\rangle$. Find the velocity and speed of the particle when $t = 0$.

$$r'(t) = \left\langle \frac{(-\sin t)(e^t) - (\cos t)e^t}{e^{2t}}, \frac{(\cos t)e^t - (\sin t)e^t}{e^{2t}} \right\rangle$$

$r'(0) = \langle -1, 1 \rangle$ ← velocity

speed $= |\langle -1, 1 \rangle|$
$= \sqrt{2}$

18. $f(x) = \begin{cases} bx^2 - 2ax + 5 & \text{if } x \le 2 \\ ax - 6 & \text{if } x > 2 \end{cases}$,

a.) What must be true for $f(x)$ to be continuous everywhere?

b.) Find the values of a and b that make $f(x)$ differentiable everywhere, if possible. If not possible, explain why.

a) continuous: equate pieces of $f(x)$ at $x=2$

$b(4) - 2a(2) + 5 = a(2) - 6$

$\boxed{4b + 11 = 6a}$

b) differentiable: equate pieces of $f'(x)$ at $x=2$

$f'(x) = \begin{cases} 2bx - 2a & \text{if } x \le 2 \\ a & \text{if } x > 2 \end{cases}$

$2b(2) - 2a = a \rightarrow 4b = 3a \rightarrow \boxed{a = \frac{4}{3}b}$

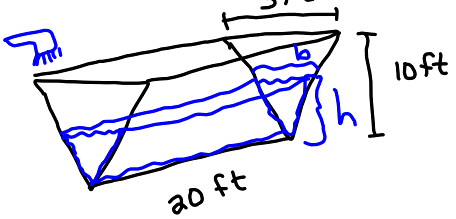
$4b + 11 = 6\left(\frac{4}{3}b\right)$

$4b + 11 = 8b \rightarrow 11 = 4b$

$\boxed{b = \frac{11}{4}}$

$a = \frac{4}{3}\left(\frac{11}{4}\right)$ $\boxed{a = \frac{11}{3}}$

19. A trough is 20 feet long. The end of the trough is an isosceles triangle with height 10 feet and length of 3 feet across the top. If water is poured in the trough at a rate of 3 cubic feet per minute, how fast is the water level rising when the height of the water is 1 foot?



Given: $\frac{dV}{dt} = 3 \frac{ft^3}{min}$

Find: $\left.\frac{dh}{dt}\right|_{h=1\ ft}$

$\frac{3}{10} = \frac{b}{h}$

$b = \frac{3}{10}h$

$V = \left(\frac{1}{2}bh\right)(20)$

$V = 10bh$

$V = 10 \cdot \left(\frac{3}{10}h\right)h$

$V = 3h^2$

$\frac{dV}{dt} = 6h\frac{dh}{dt}$

$3 = 6\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{2}\frac{ft}{min}$

20. Given the equation $2e^{xy} = x + y$, find $\frac{dy}{dx}$ when $x = 0$ and $y = 2$.

$2(xy)'e^{xy} = (x+y)'$

$2\left(y + x\frac{dy}{dx}\right)e^{xy} = 1 + \frac{dy}{dx}$

let $x = 0$, $y = 2$

$2\left(2 + 0\frac{dy}{dx}\right)e^0 = 1 + \frac{dy}{dx}$

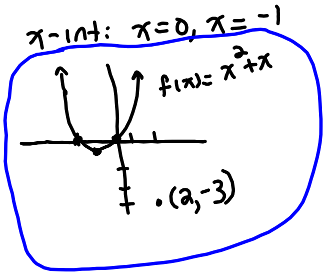
$4 = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = 3$

21. Sketch the graph of $f(x) = x^2 + x$ and show there are two tangent lines to the graph of $f(x)$ that pass through the point $(2, -3)$. Find an equation of these tangent lines.

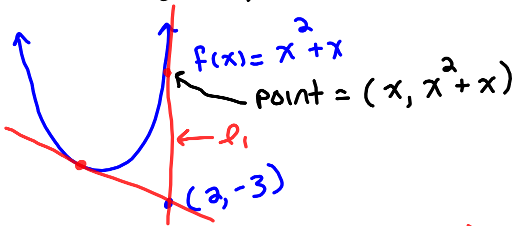
$f(x) = x^2 + x$
$= x(x+1)$

x-int: $x = 0, x = -1$

vertex: $f'(x) = 0$
$2x + 1 = 0$
$x = -\frac{1}{2}$
$y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$



blow up



slope of ℓ_1 is $m = \frac{\text{rise}}{\text{run}} = \frac{x^2 + x - (-3)}{x-2}$

$m = \frac{x^2 + x + 3}{x - 2}$

slope of ℓ_1 is also $m = 2x + 1$

equate!

$\frac{x^2 + x + 3}{x - 2} = 2x + 1$

$x^2 + x + 3 = (2x+1)(x-2)$

$x^2 + x + 3 = 2x^2 - 3x - 2$

$0 = x^2 - 4x - 5$

$0 = (x-5)(x+1)$

$x = 5$
$x = -1$

$f(x) = x^2 + x$
$f'(x) = 2x + 1$

$x = 5 \rightarrow m = f'(5)$
$m = 11$, point $= (5, f(5))$
$= (5, 30)$

equation: $y - 30 = 11(x - 5)$

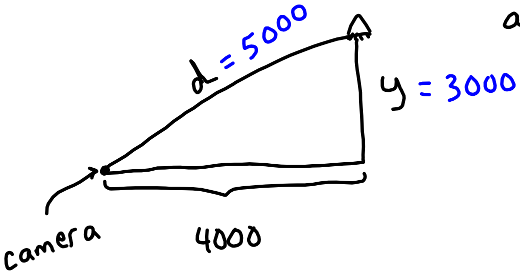
$x = -1 \rightarrow m = f'(-1)$
$m = -1$
point $= (-1, f(-1))$
$= (-1, 0)$

equation: $y - 0 = -1(x + 1)$

22. A camera is positioned 4000 feet from the base of a rocket launching pad. At a particular moment, the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

a) How fast is the distance from the camera to the rocket changing at that moment?

b) If the camera is focused on the rocket, how fast is the camera's angle of elevation changing at that moment?



a) Given $\frac{dy}{dt} = 600 \frac{f}{s}$ only when $y = 3000$

Find $\frac{dd}{dt}$ when $\frac{dy}{dt} = 600$ & $y = 3000$

$d^2 = y^2 + (4000)^2$

now fix $\frac{dy}{dt} = 600$, $y = 3000$

$d = ?$

$2d \frac{dd}{dt} = 2y \frac{dy}{dt}$

$5000 \frac{dd}{dt} = (3000)(600)$

$$\boxed{\frac{dd}{dt} = \frac{(3000)(600)}{5000} \frac{f}{s}}$$

22. A camera is positioned 4000 feet from the base of a rocket launching pad. At a particular moment, the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

a) How fast is the distance from the camera to the rocket changing at that moment?

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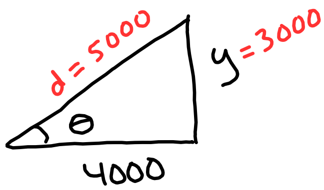

given $\frac{dy}{dt} = 600$ when $y = 3000$

Find $\frac{d\theta}{dt}$ at this moment

$\tan\theta = \frac{y}{4000}$

$\sec\theta = \frac{hyp}{adj} = \frac{5000}{4000} = \frac{5}{4}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{dy/dt}{4000}$

$\left(\frac{5}{4}\right)^2 \frac{d\theta}{dt} = \frac{600}{4000}$

so $$\boxed{\frac{d\theta}{dt} = \frac{600}{4000} \cdot \frac{16}{25} \frac{rad}{s}}$$