Section 3.2

1. Find the derivative of the following functions:

1. $\frac{d}{dx}(c) = 0$

   c any constant

2. $\frac{d}{dx} x^n = n x^{n-1}$

a.) $f(x) = 5x^5 - 7x^2 + x + 1$

$f'(x) = 25x^4 - 14x + 1$

b.) $f(x) = (x^2 - x)(x - 2)$

$f(x) = x^3 - 2x^2 - x^2 + 2x$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

c.) $f(x) = x^5 + \sqrt{x} - \dfrac{3}{x^2}$

$f(x) = x^5 + x^{\frac{1}{2}} - 3x^{-2}$

$f'(x) = 5x^4 + \frac{1}{2}x^{-\frac{1}{2}} + 6x^{-3}$

$f'(x) = 5x^4 + \dfrac{1}{2\sqrt{x}} + \dfrac{6}{x^3}$

d.) $f(t) = \dfrac{1+t^2-\sqrt[3]{t}}{t^2}$

$$f(t) = \frac{1}{t^2} + \frac{t^2}{t^2} - \frac{\sqrt[3]{t}}{t^2}$$

$$\frac{t^{\frac{1}{3}}}{t^2} = t^{\frac{1}{3}-2} = t^{-\frac{5}{3}}$$

$$f(t) = t^{-2} + 1 - t^{-\frac{5}{3}-\frac{3}{3}}$$

$$\boxed{f'(t) = -2t^{-3} + \frac{5}{3}t^{-\frac{8}{3}}}$$

$$f'(t) = \frac{-2}{t^3} + \frac{5}{3t^{8/3}} = \frac{-2}{t^3} + \frac{5}{3\sqrt[3]{t^8}}$$

Product Rule:

$f(x) = g(x) \cdot h(x)$

$f'(x) = g(x)h'(x) + g'(x)h(x)$

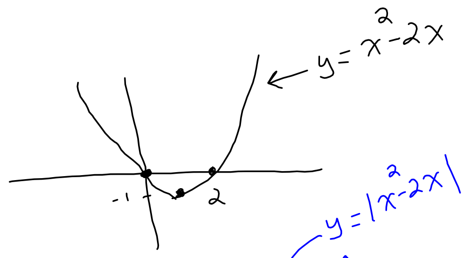
Quotient Rule:

$f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

e.) $g(x) = \dfrac{x^2+x-4}{2x-x^3}$

$$g'(x) = \frac{(2x+1)(2x-x^3) - (x^2+x-4)(2-3x^2)}{(2x-x^3)^2}$$



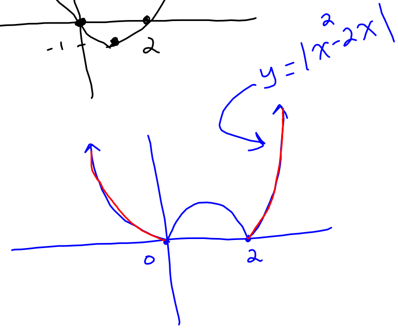
f.) $f(x) = |\underbrace{x^2-2x}|$

$* \; y = x^2 - 2x$

$y = x(x-2)$

vertex: $y' = 0 \rightarrow 2x - 2 = 0$

$x = 1, \; y = -1$



$$|x^2-2x| = \begin{cases} x^2-2x & \text{if } x \le 0, \; x \ge 2 \\ -x^2+2x & \text{if } 0 < x < 2 \end{cases}$$

$$f'(x) = \begin{cases} 2x-2 & \text{if } x<0, \; x>2 \\ -2x+2 & \text{if } 0<x<2 \end{cases}$$

$f(x)$ not differentiable at $x=0$ & $x=2$ (sharp turns)

2. Given $h = f(x)g(x)$, $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, $f(3) = -2$
$f'(6) = 7$. Find $h'(3)$.

missing info added!

$h = f(x)g(x) \leftarrow$ product Rule

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h'(3) = \underbrace{f(3)}_{-2}\underbrace{g'(3)}_{4} + \underbrace{f'(3)}_{2}\underbrace{g(3)}_{6} = -8 + 12$

$= \boxed{4}$

3. Given $h = \dfrac{f(x)}{g(x)}$, $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, $f(3) = -2$
$f'(6) = 7$. Find $h'(3)$.

missing info added

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(3) = \frac{\overset{2}{f'(3)}\,\overset{6}{g(3)} - \overset{-2}{f(3)}\,\overset{4}{g'(3)}}{[\underset{6}{g(3)}]^2} = \frac{12+8}{36} = \boxed{\frac{20}{36} = \frac{5}{9}}$$

4. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent lines are horizontal, if any. If there are none, support your answer. ↳ slope = 0

solve $y' = 0$

$3x^2 - 2x - 1 = 0 \rightarrow (3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$, $x = 1$

$x = -\frac{1}{3}$, $y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1 = -\frac{1}{27} - \frac{3}{27} + \frac{9}{27} + \frac{27}{27}$

$= \frac{32}{27}$ $\boxed{\text{point: } \left(-\frac{1}{3}, \frac{32}{27}\right)}$

$x = 1$, $y = 1 - 1 - 1 + 1 = 0$ $\boxed{\text{point: } (1, 0)}$

5. Find the points on the curve $y = 8x^3 + 5x + 1$ where the tangent line has slope, 1, if any. If there are none, support your answer. ↳ slope = 1

$y' = 1$

$24x^2 + 5 = 1$

$24x^2 = -4$

$x^2 = -\frac{1}{6}$ $\boxed{\text{no solution}}$

6. Find the equation of the tangent line to the graph of $f(x) = \dfrac{x^2}{x-4}$ at the point $\left(1, -\dfrac{1}{3}\right)$

① $m = f'(1)$

$$f'(x) = \frac{2x(x-4) - x^2(1)}{(x-4)^2}$$

$$f'(x) = \frac{2x^2 - 8x - x^2}{(x-4)^2}$$

$$f'(x) = \frac{x^2 - 8x}{(x-4)^2}$$

$m = f'(1)$

$\boxed{m = -\frac{7}{9}}$

equation:

$$\boxed{y + \tfrac{1}{3} = -\tfrac{7}{9}(x-1)}$$

7. Find the equation of both lines through the point $P(2, -3)$ that are tangent to the parabola $y = x^2 + 2x$.

graph $y = x^2 + 2x$

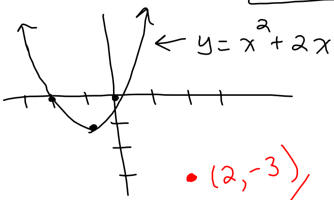
$= x(x+2)$

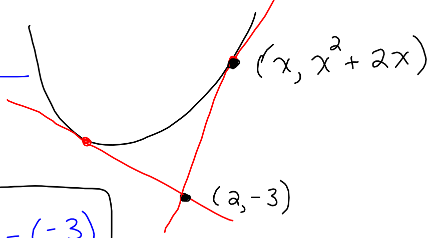
vertex: $y' = 0$

$2x + 2 = 0$

$x = -1$

$y = -1$





$y = x^2 + 2x$

$\boxed{m = 2x + 2}$

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \boxed{\frac{x^2 + 2x - (-3)}{x - 2}}$$

solve $\quad 2x + 2 = \dfrac{x^2 + 2x + 3}{x - 2}$

$$(2x+2)(x-2) = x^2 + 2x + 3$$

$$2x^2 - 4x + 2x - 4 = x^2 + 2x + 3$$

$$2x^2 - 2x - 4 = x^2 + 2x + 3$$

$$x^2 - 4x - 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(-7)}}{2}$$

$$x = \frac{4 \pm \sqrt{44}}{2}$$

② Find the equation of the tangent line to $f(x) = x^2 + 2x$ at $x = \dfrac{4 \pm \sqrt{44}}{2}$

8. At what point on the curve $y = x\sqrt{x}$ is the tangent line parallel to the line $3x - y + 6 = 0$?

$y = 3x + 6$

$m = 3$

solve $y' = 3$, where $y = x\sqrt{x} = x \cdot x^{\frac{1}{2}}$

$y = x^{\frac{3}{2}}$

$y' = \frac{3}{2} x^{\frac{1}{2}}$

solve $y' = 3$

$\frac{3}{2} x^{\frac{1}{2}} = 3$

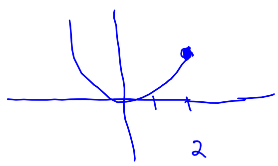
$x^{\frac{1}{2}} = 3 \cdot \frac{2}{3}$

$\sqrt{x} = 2 \rightarrow x = 4 \rightarrow y = 4\sqrt{4}$

$y = 8$

$\boxed{\text{point is } (4, 8)}$

9. If $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$, find the value of m and b that make $f(x)$ differentiable everywhere.



Force continuity:

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$\lim_{x \to 2^+} (mx + b) = \lim_{x \to 2^-} (x^2)$

① $\boxed{2m + b = 4}$

$m = 4$

$8 + b = 4$

$\boxed{b = -4}$

Force differentiability

$\lim_{x \to 2^+} f'(x) = \lim_{x \to 2^-} f'(x)$

$\lim_{x \to 2^+} m = \lim_{x \to 2^-} (2x)$

$\boxed{m = 4}$

10. A particle moves according to the equation of motion

$s(t) = 4t^3 - 9t^2 + 6t + 2$, where $s(t)$ is measured in meters and t in seconds.

(a) Find the velocity at time t.

$$\boxed{v(t) = s'(t) = 12t^2 - 18t + 6}$$

(b) When is the particle at rest?

$$v(t) = 0 \longrightarrow 12t^2 - 18t + 6 = 0$$

$$6(2t^2 - 3t + 1) = 0 \rightarrow 6(2t-1)(t-1) = 0$$

$$\boxed{t = \frac{1}{2}\ s \qquad t = 1\ s}$$

(c) When is the particle moving in the positive direction?

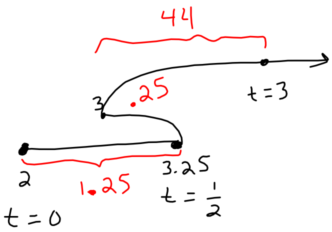
solve $v(t) > 0$

$v(t) = 6(2t-1)(t-1)$

$v(t) > 0$ on

$[0, \frac{1}{2}) \cup (1, \infty)$

(d) Draw a diagram that represents the motion of the particle.



$s(t) = 4t^3 - 9t^2 + 6t + 2$

$s(0) = 2$

$s(\frac{1}{2}) = 3.25$

$s(1) = 4 - 9 + 6 + 2$

$s(1) = 3$

$s(3) = 47$

(e) Find the distance traveled in the first $\boxed{3}$ seconds.

Total distance traveled in first 3 seconds is

$$s(\tfrac{1}{2}) - s(0) + \left| s(1) - s(\tfrac{1}{2}) \right| + s(3) - s(1)$$

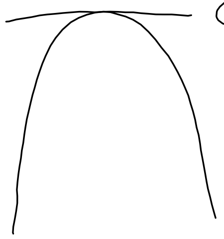
$$3.25 - 2 + |3 - 3.25| + 47 - 3$$

$$= 1.25 + .25 + 44$$

$$= 1.5 + 44$$

$$= \boxed{45.5\ m}$$

11. A ball is thrown vertically upward with a velocity of 80 feet per second. The height after t seconds is given by $h(t) = 80t - 16t^2$. What is the maximum height of the ball?



① solve $v(t) = h'(t) = 0$

$$80 - 32t = 0$$

$$32t = 80$$

$$t = \frac{80}{32}$$

$$t = 2.5$$

$$h(2.5) = 80(2.5) - 16(2.5)^2 = 100 \text{ feet}$$

Two special limits:

$$\boxed{(1)\ \lim_{x\to 0}\frac{\sin x}{x}=1}\qquad (2)\ \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

Section 3.4

12. Compute the following limits:

a.) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{5x} = 3\cdot\frac{1}{5}\lim_{x\to 0}\frac{\sin(3x)}{3x} \to 1$

$$= \boxed{\frac{3}{5}}$$

b.) $\displaystyle\lim_{x\to 0}\frac{\sin(7x)}{\sin(5x)} = \lim_{x\to 0}\frac{\frac{\sin(7x)}{7x}\cdot 7}{\frac{\sin(5x)}{5x}\cdot 5} = \boxed{\frac{7}{5}}$

c.) $\displaystyle\lim_{x\to 0}\frac{\sin^2 6x}{x^2} = \lim_{x\to 0}\frac{\sin(6x)}{6x}\,\frac{\sin(6x)}{6x}\cdot 6\cdot 6$

$$= \boxed{36}$$

Aside

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$\frac{\sin x}{x} \to 1$, $\frac{1}{\cos x} \to \frac{1}{\cos(0)} = 1$

$= 1$

New Rule!

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

d.) $\displaystyle \lim_{x\to 0} \frac{\tan x}{4x} = \frac{1}{4} \lim_{x\to 0} \frac{\tan x}{x} = \boxed{\frac{1}{4}}$

e.) $\displaystyle \lim_{x\to 0} \frac{\sin 8x}{\tan(5x)} = \lim_{x\to 0} \frac{\frac{\sin(8x)}{8x} \cdot 8}{\frac{\tan(5x)}{5x} \cdot 5} = \frac{8}{5}$

f.) $\displaystyle \lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}} = \boxed{0}$

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$

13. Find the derivative of the following functions:

a.) $f(x) = \dfrac{\sin x}{1 + \cos x}$

$$f'(x) = \frac{(\cos x)(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$f'(x) = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

1

b.) $y = \sec x - 5\tan x$

$$\boxed{y' = \sec x \tan x - 5\sec^2 x}$$

14. Find $f'(\frac{\pi}{6})$ for $f(x) = -2\cot x$

$$f'(x) = -2(-\csc^2 x)$$

$$f'(x) = 2\csc^2 x$$

$$f'(\tfrac{\pi}{6}) = 2\csc^2(\tfrac{\pi}{6})$$

$$= 2(2)^2 = \boxed{8}$$

$$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

15. Find the tangent line to the graph of

$f(x) = \sec x - 2\cos x$ where $x = \frac{\pi}{3}$.

$m = f'(\frac{\pi}{3})$

$f'(x) = \sec x \tan x + 2\sin x$

$f'(\frac{\pi}{3}) = \sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3}$

$= 2 \cdot \sqrt{3} + 2 \cdot \frac{\sqrt{3}}{2}$

$= 2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$

$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

$\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$m = 3\sqrt{3}$

Point: $x = \frac{\pi}{3}$, $y = \sec\frac{\pi}{3} - 2\cos\frac{\pi}{3}$

$y = \frac{1}{\frac{1}{2}} - 2 \cdot \frac{1}{2}$

$y = 2 - 1$

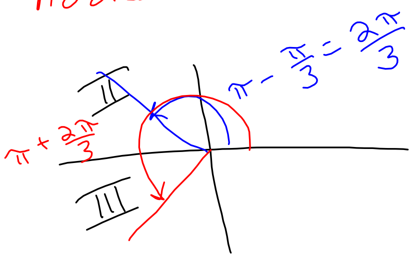
$y = 1$

$m = 3\sqrt{3}$, Point $(\frac{\pi}{3}, 1)$

$$\boxed{y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)}$$

EXAMPLE 4: For what value(s) of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

Added problem!



$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

solve $f'(x) = 0$

$1 + 2\cos x = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

all Q II angles $\longrightarrow x = \frac{2\pi}{3} + 2n\pi$, n = integer

all Q III angles $\longrightarrow x = \frac{5\pi}{3} + 2n\pi$, n = integer