Chain Rule: We use the chain rule when we are differentiating a function written as a composition of functions, that is $\underline{f(x) = g(h(x))}$. Then $f'(x) = \underline{\underline{g'}}(\underline{\underline{h(x)}})\underline{\underline{h'(x)}}$.

$f(x) = g \circ h$

Generalized Power Rule: If $\underline{f(x) = (g(x))^n}$, then $\underline{f'(x) = n\,\underline{(g(x))}^{n-1}\,\underline{g'(x)}}$

Section 3.5

1. Find the derivative of the following functions:

   a.) $f(x) = (x^3 + x + 1)^8$

   $f'(x) = 8(x^3 + x + 1)^7 \cdot (3x^2 + 1)$

   b.) $f(x) = \sqrt{x^5 - \frac{3}{x^2} + \sin(x) - \sec(x)}$ $= (x^5 - 3x^{-2} + \sin x - \sec x)^{\frac{1}{2}}$

   $f'(x) = \frac{1}{2}(x^5 - 3x^{-2} + \sin x - \sec x)^{-\frac{1}{2}}(5x^4 + 6x^{-3} + \cos x - \sec x \tan x)$

   c.) $f(x) = \frac{1}{(x^2 + x - 1)^2}$ $= (x^2 + x - 1)^{-2}$

   $f'(x) = -2(x^2 + x - 1)^{-3}(2x + 1)$

d.) $h(x) = \tan(x^2)$ ← composition!

$h'(x) = \sec^2(x^2) \cdot 2x$

e.) $g(x) = \cos^3(x^2 + a^2) = \left(\cos(x^2 + a^2)\right)^3$

a^2 is constant so its derivative is zero

$g'(x) = 3\left(\cos(x^2+a^2)\right)^2 \cdot \left(-\sin\left(x^2+a^2\right)(2x)\right)$

f.) $g(x) = \underset{\left(\sin(x^2)\right)^3}{\underbrace{\sin^3(x^2)}} + \cot(\sin(2x))$

$g(x) = 3\sin^2(x^2) \cdot \cos(x^2) \cdot (2x) - \csc^2(\sin(2x)) \cdot \cos(2x) \cdot 2$

g.) $f(x) = \underbrace{(2x+1)^5}_{g}\underbrace{(\sqrt{x}-x+3)^7}_{h}$

$f'(x) = gh' + g'h$

$f'(x) = \underbrace{(2x+1)^5}_{g}\underbrace{7(\sqrt{x}-x+3)^6\left(\frac{1}{2}x^{-\frac{1}{2}}-1\right)}_{h'} + \underbrace{5(2x+1)^4(2)}_{g'}\underbrace{(\sqrt{x}-x+3)^7}_{h}$

h.) $h(x) = \frac{x}{(x^5+1)^4}$ $\quad \frac{f}{g}$

$f = x$

$g = (x^5+1)^4$

$h'(x) = \frac{f'g - fg'}{g^2}$

$$\boxed{h'(x) = \frac{(1)(x^5+1)^4 - x\cdot 4(x^5+1)^3(5x^4)}{(x^5+1)^8}}$$

also, could have used product rule for $h(x) = x(x^5+1)^{-4}$

$\rightarrow h(x) = x(x^5+1)^{-4}$

OR //

$$\boxed{h'(x) = x\left[-4(x^5+1)^{-5}\cdot 5x^4\right] + (1)(x^5+1)^{-4}}$$

2. Given $h = f \circ g$, $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, $f'(6) = 7$. Find $h'(3)$.

$h(x) = f \circ g$

$h(x) = \underline{\underline{f(g(x))}}$ composition!

$h'(x) = f'(g(x))\, g'(x)$

$h'(3) = f'(\underbrace{g(3)}_{6})\,\underbrace{g'(3)}_{4} = \overset{7}{f'(6)}\cdot 4$

$= 7 \cdot 4 = \boxed{28}$

3. Suppose that $F(x) = f(x^4)$ and $G(x) = (f(x))^4$. Also, suppose it is given that $f(2) = -1$, $f(16) = 3$, $f'(2) = -2$ and $f'(16) = 4$. Compute $F'(2)$ and $G'(2)$.

① $F(x) = f(x^4)$

$F'(x) = f'(x^4) \cdot 4x^3$

$F'(2) = \underbrace{f'(16)}_{4} \cdot 4(2)^3$

$= 4 \cdot 32 = \boxed{128}$

② $G(x) = (f(x))^4$

$G'(x) = 4(f(x))^3 \cdot f'(x)$

$G'(2) = 4(\underbrace{f(2)}_{-1})^3\, \underbrace{f'(2)}_{-2}$

$= 4(-1)^3(-2) = \boxed{8}$

4. If $G(t) = (t + f(\tan 2t))^3$, find an expression for $G'(t)$.

$G'(t) = 3(t + \underbrace{f(\tan 2t)}{})^2 \left(1 + f'(\tan 2t) \cdot \sec^2 2t\,(2)\right)$

Section 3.6

5. $\boxed{\text{Find } \frac{dy}{dx}}$ if $\underbrace{x^4 - 4x^2y^2 + y^3 = 0}$

Aside:

$$y = x^2 + 3$$

$$\frac{dy}{dx} = 2x$$

$$\frac{d}{dx} y^2 = 2y\frac{dy}{dx}$$

$$x^4 - \underbrace{4x^2y^2} + y^3 = 0$$

$$4x^3 - \left[4x^2 \cdot 2y\frac{dy}{dx} + 8xy^2\right] + 3y^2\frac{dy}{dx} = 0$$

$$\underbrace{4x^3}_{\hookrightarrow} - \underline{8x^2y}\,\frac{dy}{dx} \underbrace{-8xy^2}_{\hookrightarrow} + \underline{3y^2}\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-8x^2y + 3y^2\right) = -4x^3 + 8xy^2$$

$$\boxed{\frac{dy}{dx} = \frac{-4x^3 + 8xy^2}{-8x^2y + 3y^2}}$$

6. Find $\frac{dy}{dx}$ for $\cos(2x) - \sin(x + y) = 1$

$$\underbrace{-\sin(2x)\cdot 2}_{\hookrightarrow} - \cos(x+y)\left(1 + \frac{dy}{dx}\right) = 0$$

$$-\cos(x+y)\left(1 + \frac{dy}{dx}\right) = 2\sin(2x)$$

$$1 + \frac{dy}{dx} = \frac{2\sin(2x)}{-\cos(x+y)}$$

$$\boxed{\frac{dy}{dx} = \frac{2\sin(2x)}{-\cos(x+y)} - 1}$$

7. Find the equation of the line tangent to $x^2 + y^2 = 2$ at (1,1).

First find $\frac{dy}{dx}$:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$$

$$\boxed{\frac{dy}{dx} = \frac{-x_{=1}}{y_{=1}}}$$

Tangent line :

point = (1,1)

slope = $\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}}$

$m = \frac{-1}{1} = -1$

$y - 1 = -1(x-1)$

$y - 1 = -x + 1$

$\boxed{y = -x + 2}$

8. Regard y as the independent variable and x as the dependent variable, and use implicit differentiation to find $\frac{dx}{dy}$ for the equation $\underbrace{(x^2+y^2)^2}$ $= 2x^2y$.

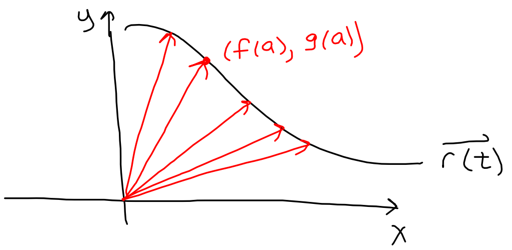
$x = f(y)$

$$\underbrace{2(x^2+y^2)}\left(\underbrace{2x\frac{dx}{dy}} + \underbrace{2y}\right) = 2x^2(1) + 4x\frac{dx}{dy}\cdot y$$

$$\underline{4x(x^2+y^2)}\frac{dx}{dy} + \underline{4y(x^2+y^2)} = 2x^2 + \underline{4xy}\frac{dx}{dy}$$

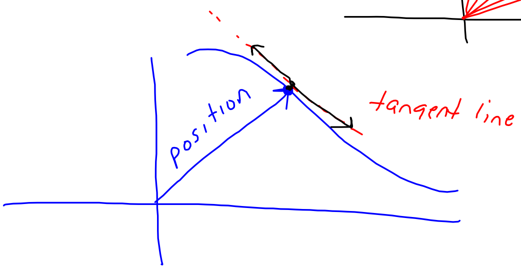
$$\frac{dx}{dy}\left(4x(x^2+y^2) - 4xy\right) = -4y(x^2+y^2) + 2x^2$$

$$\boxed{\frac{dx}{dy} = \frac{-4y(x^2+y^2) + 2x^2}{4x(x^2+y^2) - 4xy}}$$



$r(t) = \langle f(t), g(t) \rangle$

$r(a) = \langle \underbrace{f(a)}_{x}, \underbrace{g(a)}_{y} \rangle$



Section 3.7

9. Find the angle between the tangent vector and the position vector for $\mathbf{r(t)} = \langle t^2, 2t^3 \rangle$ at the point where $t = -1$.

position vector at $t=-1$ is $\overline{r(-1)}$

tangent vector at $t=-1$ is $r'(-1)$

$r(t) = \langle t^2, 2t^3 \rangle$ so $r(-1) = \langle 1, -2 \rangle$

$r'(t) = \langle 2t, 6t^2 \rangle$ so $r'(-1) = \langle -2, 6 \rangle$

$$\cos\theta = \frac{\langle 1,-2\rangle \cdot \langle -2,6\rangle}{|\langle 1,-2\rangle| \cdot |\langle -2,6\rangle|}$$

$$\cos\theta = \frac{-2-12}{\sqrt{5}\sqrt{40}}$$

$$\cos\theta = \frac{-14}{\sqrt{200}}$$

$$\cos\theta = \frac{-14}{10\sqrt{2}}$$

$$\boxed{\theta = \arccos\left(\frac{-7}{5\sqrt{2}}\right)}$$

10. Find the vector and parametric equations of the line tangent to $\mathbf{r(t)} = \langle t^3 + 2t, 4t - 5 \rangle$ at the point where $t = 2$.

section 1.3

In general, the vector equation of the line passing thru $\vec{r}_0$ & parallel to $\vec{v}$ is $\boxed{r(t) = \vec{r}_0 + t\vec{v}}$

Here, $\vec{r}_0 = \overline{r(2)} = \langle 2^3 + 2(2), 4(2) - 5 \rangle$

$\boxed{\vec{r}_0 = \langle 12, 3 \rangle}$

$r(t) = \langle t^3 + 2t, 4t - 5 \rangle$

$r'(t) = \langle 3t^2 + 2, 4 \rangle$

$r'(2) = \langle 14, 4 \rangle$

And, $\vec{v} = \overline{r'(2)}$

$\boxed{\vec{v} = \langle 14, 4 \rangle}$

vector equation of tangent line is $\vec{r}_0 + t\vec{v}$

$= \langle 12, 3 \rangle + t\langle 14, 4 \rangle$

$= \langle 12 + 14t, 3 + 4t \rangle$

Parametric equations of tangent line are

$x = 12 + 14t$

$y = 3 + 4t$

11. Sketch the curve $\mathbf{r}(\mathbf{t}) = \langle t^2, t\rangle$. Find the tangent and unit tangent vector to the curve at the point (4, 2). Draw the position and tangent vector along with the sketch of the curve at the point (4, 2).

note: $r(\underline{\underline{2}}) = \langle 4, 2\rangle$

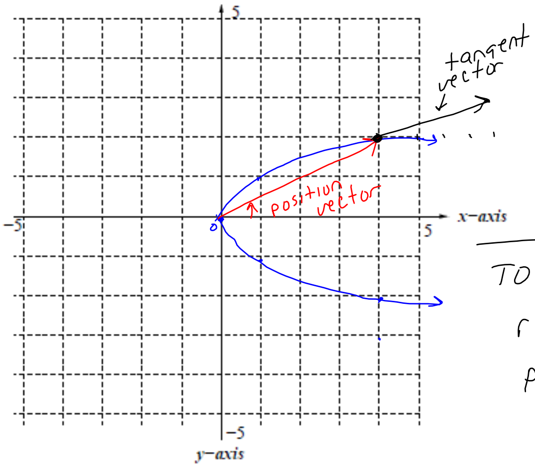
tangent vector at $(4, 2)$ is $r'(2)$

$r'(t) = \langle 2t, 1\rangle$

$r'(2) = \langle 4, 1\rangle \leftarrow$ tangent vector

unit tangent vector

$$\vec{u} = \frac{\langle 4,1\rangle}{|\langle 4,1\rangle|} = \frac{\langle 4,1\rangle}{\sqrt{17}} = \left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$$



To sketch the curve $r(t) = \langle t^2, t\rangle$ eliminate the parameter to get a cartesian equation.

$x = t^2$

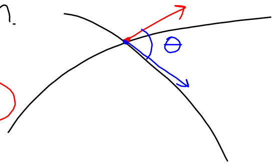
$y = t$

$\boxed{x = y^2}$

12. Find the angle of intersection of the curves

$\mathbf{r_1(s)} = \langle s - 2, s^2\rangle$ and $\mathbf{r_2(t)} = \langle 1 - t, 3 + t^2\rangle$

the angle of intersection of two curves is defined to be the angle between the tangent vectors at the point of intersection.



intersection: solve $s - 2 = 1 - t \rightarrow \boxed{s = 3 - t}$

$s^2 = 3 + t^2$

$(3 - t)^2 = 3 + t^2$

$9 - 6t + \cancel{t^2} = 3 + \cancel{t^2}$

$6 = 6t$

$\boxed{t = 1}$

$\boxed{s = 2}$

Tangent vectors :

$r_1'(s) = \langle 1, 2s\rangle$ so $r_1'(2) = \langle 1, 4\rangle$

$r_2'(t) = \langle -1, 2t\rangle$ $\quad r_2'(1) = \langle -1, 2\rangle$ $\Big\} \leftarrow$ tangent vectors

Angle of intersection $\cos\theta = \dfrac{\langle 1, 4\rangle \cdot \langle -1, 2\rangle}{\sqrt{17}\ \sqrt{5}}$

$\cos\theta = \dfrac{7}{\sqrt{85}}$

$\theta = \arccos\left(\dfrac{7}{\sqrt{85}}\right)$

Section 3.8

13. Find y'' for $y = \sqrt{x^2+1}$. $= (x^2+1)^{\frac{1}{2}}$

$$y' = \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x)$$

$$y' = (x^2+1)^{-\frac{1}{2}} \cdot x \leftarrow \text{product rule!}$$

$$\boxed{y'' = (x^2+1)^{-\frac{1}{2}}(1) + -\frac{1}{2}(x^2+1)^{-\frac{3}{2}}(2x)\cdot x}$$

14. If $\mathbf{r}(\mathbf{t}) = \langle t^3, t^2\rangle$ represents the position of a particle at time t, find the angle between the velocity and the acceleration vector at time $t = 1$.

velocity at $t=1$ is $r'(1)$

acceleration at $t=1$ is $r''(1)$

$r'(t) = \langle 3t^2, 2t\rangle$

$r'(1) = \langle 3, 2\rangle$

$r''(t) = \langle 6t, 2\rangle$

$r''(1) = \langle 6, 2\rangle$

$$\cos\theta = \frac{\langle 3,2\rangle\cdot\langle 6,2\rangle}{\sqrt{13}\ \sqrt{40}}$$

$$\cos\theta = \frac{18+4}{\sqrt{13}\ \sqrt{40}}$$

$$\boxed{\theta = \arccos\left(\frac{22}{\sqrt{13}\ \sqrt{40}}\right)}$$

15. Find the 98*th* derivative of:

a.) $f(x) = \frac{1}{x^2}$

$f(x) = x^{-2}$

$f'(x) = -\underline{2}x^{-3}$

$f''(x) = \underline{3 \cdot 2}\, x^{-4}$

$f'''(x) = -\underline{4 \cdot 3 \cdot 2}\, x^{-5}$

$f^{4}(x) = 5 \cdot 4 \cdot 3 \cdot 2\, x^{-\underline{\underline{6}}}$

$\vdots$

$f^{n}(x) = (-1)^{n}(n+1)!\, x^{-(n+2)}$

$f^{98}(x) = (-1)^{98}\, 99!\, x^{-100}$

$$\boxed{f^{98}(x) = \frac{99!}{x^{100}}}$$

b.) $f(x) = \sin(3x)$

$f'(x) = 3\cos(3x)$

$f''(x) = -3^{2}\sin(3x)$

$f'''(x) = -3^{3}\cos(3x)$

$f^{\underline{\underline{4}}}(x) = 3^{\underline{\underline{4}}}\sin(3x)$

every four derivatives, you are back at $\sin(3x)$

96 is divisible by four, so $f^{96}(x) = 3^{96}\sin(3x)$

$f^{97}(x) = 3^{97}\cos(3x)$

$$\boxed{f^{98}(x) = -3^{98}\sin(3x)}$$