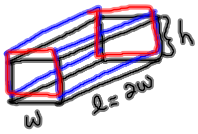
Section 5.5

1. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

① picture



② what is the constraint?

$$V = 10\ m^3 \Rightarrow \boxed{2w^2h = 10}$$

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

③ what do we want to do?

minimize the cost $C = C_{base} + C_{sides}$

$$= \$10\underbrace{(2w^2)}_{\text{base}} + \$6\Big(\underbrace{2\cdot 2wh}_{\text{sides}} + \underbrace{2\cdot hw}_{\text{front \& back}}\Big)$$

6wh

$$C = 20w^2 + 36wh$$

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right)$$

$$\boxed{C = 20w^2 + \frac{180}{w}} = 20w^2 + 180w^{-1}$$

④ Find critical numbers

$$C' = 40w - \frac{180}{w^2}$$

$$C' = \frac{40w^3 - 180}{w^2}$$

cn: solve $C' = 0$

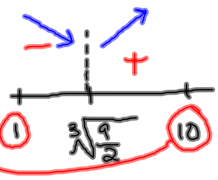
cn: $40w^3 - 180 = 0$

$$40w^3 = 180$$

$$w^3 = \frac{180}{40} = \frac{9}{2}$$

$$\boxed{w = \sqrt[3]{\frac{9}{2}}}$$

prove $w = \sqrt[3]{\frac{9}{2}}$ to prove a minimum occurs.



proves minimum.

Total cost: $C = 20w^2 + \frac{180}{w}$

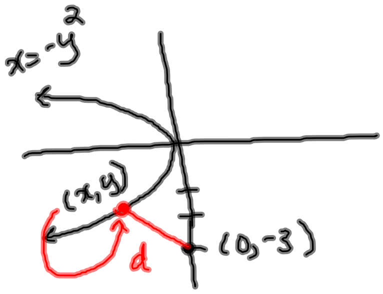
$$C\left(\sqrt[3]{\tfrac{9}{2}}\right) = \$\left[20\left(\sqrt[3]{\tfrac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{\frac{9}{2}}}\right]$$

$$\approx \boxed{\$\,163.54}$$

2. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

① picture $\quad x+y^2=0$

$x=-y^2$



minimize d.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$x=-y^2$

$$d=\sqrt{x^2+(y+3)^2}$$

$(-y^2)^2=y^4$

$$d=\sqrt{y^4+(y+3)^2}$$

$$d^2=y^4+(y+3)^2$$

$$2d\,d'=4y^3+2(y+3)$$

$$d'=\frac{4y^3+2(y+3)}{2d}$$

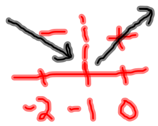
$$=\frac{4y^3+2(y+3)}{2\sqrt{y^4+(y+3)^2}}$$

cn:

$4y^3+2(y+3)=0$

$4y^3+2y+6=0$

$y=-1$



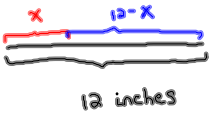
Point on parabola: $y=-1$

$x=-y^2=-1$

$\boxed{\text{Point}=(-1,-1)}$

3. A piece of wire 12 inches long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. How should the wire be cut so that the total area enclosed is a maximum? A minimum?

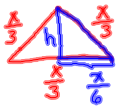
triangle circle



domain: $0 \le x \le 12$

Total area: $A = A_\Delta + A_o$

$A_\Delta = \frac{1}{2} b h$



$\boxed{b = \frac{x}{3}}$

$\therefore A_\Delta = \frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{x}{2\sqrt{3}}\right)$

$A_\Delta = \frac{x^2}{12\sqrt{3}}$

Find h: $\frac{x^2}{9} = h^2 + \frac{x^2}{36} \Rightarrow h^2 = \frac{x^2}{9} - \frac{x^2}{36} = \frac{3x^2}{36} = \frac{x^2}{12}$

$\boxed{h = \frac{x}{\sqrt{12}} = \frac{x}{2\sqrt{3}}}$

$A_o = \pi r^2$

$C = 12 - x \Rightarrow 2\pi r = 12 - x$

$\Rightarrow r = \frac{12-x}{2\pi}$

$A_o = \pi\left(\frac{12-x}{2\pi}\right)^2$

$A_o = \frac{1}{4\pi}(12-x)^2$

$A = A_\Delta + A_o$

$A = \frac{x^2}{12\sqrt{3}} + \frac{1}{4\pi}(12-x)^2$

$A' = \frac{2x}{12\sqrt{3}} + \frac{1}{4\pi} 2(12-x)(-1)$

$A' = \frac{x}{6\sqrt{3}} - \frac{12-x}{2\pi}$

$A' = 0$

$\frac{x}{6\sqrt{3}} - \frac{12-x}{2\pi} = 0$

$\frac{x}{6\sqrt{3}} = \frac{12-x}{2\pi}$

$2\pi x = 6\sqrt{3}(12-x)$

$2\pi x = 72\sqrt{3} - 6\sqrt{3}x$

$x(2\pi + 6\sqrt{3}) = 72\sqrt{3}$

$x = \frac{72\sqrt{3}}{2\pi + 6\sqrt{3}} \approx 7.47$

evaluate the area at

$x = 0,\ x = 12,\ x = 7.47$

$A = \frac{x^2}{12\sqrt{3}} + \frac{1}{4\pi}(12-x)^2$

$A(0) = \frac{144}{4\pi} \approx 11.45$ largest

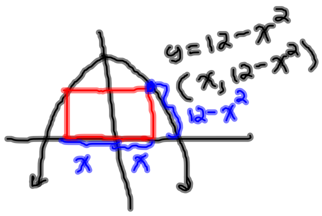
$A(12) = \frac{12}{\sqrt{3}} \approx 6.9$

$A(cn) \approx 4.318$

To make the enclosed area a minimum, cut 7.47 inches into a triangle

To make the enclosed area a maximum, cut 12 inches into a circle

4. What are the dimensions of the largest rectangle that can be inscribed in the area bounded by the curve $y = 12 - x^2$ and the x-axis?

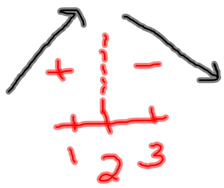


maximize the area of the rectangle

$A = (2x)(12 - x^2)$

$A = 24x - 2x^3$

$A' = 24 - 6x^2$



$A' = 0 \Rightarrow 24 - 6x^2 = 0$

$6x^2 = 24$

$x^2 = 4$ $\boxed{x = 2}$

dimensions: $2x = 4$

$y = 12 - x^2 = 12 - 4 = 8$

$\boxed{\begin{array}{l}\text{length} = 4 \\ \text{width} = 8\end{array}}$

Section 5.7

5. Given $f''(x) = 2e^x - 4\sin(x)$, $f(0) = 1$, and $f'(0) = 2$, find $f(x)$.

$f'(x) =$ antiderivative of $f''(x)$

$f'(x) = 2e^x - 4(-\cos x) + C$

$f'(0) = 2 \Rightarrow 2e^0 + 4\cos(0) + C = 2$

$6 + C = 2 \Rightarrow \boxed{C = -4}$

$f'(x) = 2e^x + 4\cos x - 4$

$f(x) =$ antiderivative of $f'(x)$

$f(x) = 2e^x + 4\sin x - 4x + K$

$f(0) = 1 \Rightarrow 2e^0 + 4\sin(0) - 4(0) + K = 1$

$2 + K = 1 \Rightarrow \boxed{K = -1}$

6. A particle accelerates according to the equation $a(t) = .12t^2 + 4$. If the initial velocity is 10 and the initial position is 0, find the position function $s(t)$.

$a(t) = .12t^2 + 4$ $\qquad v_0 = 10$

$s_0 = 0$

$v(t) = .12\frac{t^3}{3} + 4t + \cancel{v_0}^{10} \Rightarrow v(t) = .04t^3 + 4t + 10$

$s(t) = .04\frac{t^4}{4} + 4\frac{t^2}{2} + 10t + \cancel{s_0}^{0}$

$\boxed{s(t) = .01t^4 + 2t^2 + 10t}$

$v_0 = 0$ $\qquad$ $S_0 = 450$

7. A stone is dropped from a 450 meter tall building.

   a.) *Derive* a formula for the height of the stone at time t. Note the acceleration due to gravity is -9.8 meters per second squared.

   b.) With what velocity does the stone hit the ground?

$a(t) = -9.8$

$v(t) = -9.8t + \cancel{v_0} \to 0$

$v(t) = -9.8t$

$s(t) = -9.8\frac{t^2}{2} + \cancel{S_0} \to 450$

(a) $s(t) = -4.9t^2 + 450$

(b) ① when does it hit the ground?

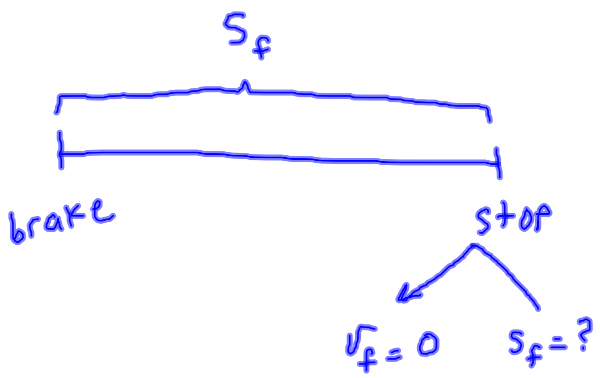
$s(t) = 0$

$-4.9t^2 + 450 = 0$

$t^2 = \frac{450}{4.9} \Rightarrow t = \sqrt{\frac{450}{4.9}}$ seconds

$v\left(\sqrt{\frac{450}{4.9}}\right) = -9.8\left(\sqrt{\frac{450}{4.9}}\right) \frac{m}{sec}$

8. A car is traveling at a speed of 220/3 feet per second when the brakes are fully applied thus producing a constant deceleration of 40 feet per second squared. How far does the car travel before coming to a stop?



$a(t) = -40, \quad v_0 = \frac{220}{3}$

$v(t) = -40t + \frac{220}{3}$

$s(t) = -40\frac{t^2}{2} + \frac{220}{3}t + \cancel{s_0}^{\,0}$

$s(t) = -20t^2 + \frac{220}{3}t$

car stops when

$v(t) = 0$

$-40t + \frac{220}{3} = 0$

$40t = \frac{220}{3}$

$t = \frac{220}{3(40)} = \frac{11}{6}$ s

$s\left(\frac{11}{6}\right) = -20\left(\frac{11}{6}\right)^2 + \frac{220}{3}\left(\frac{11}{6}\right)$ feet

≈ 67 feet

9. Find the vector functions that describe the velocity and position of a particle that has an acceleration of $\mathbf{a}(t) = \langle 0, 2 \rangle$, initial velocity of $\mathbf{v}(0) = \langle 1, -1 \rangle$ and an initial position of $\mathbf{r}(0) = \langle 0, 0 \rangle$.

$$a(t) = \langle 0, 2 \rangle$$

$$v(t) = \langle \cancel{c_1}^{\,1}, 2t + \cancel{c_2}^{\,-1} \rangle \leftarrow v_0 = \langle 1, -1 \rangle$$

$$v(t) = \langle 1, 2t - 1 \rangle$$

$$r(t) = \langle t + \cancel{c_3}^{\,0}, t^2 - t + \cancel{c_4}^{\,0} \rangle \leftarrow r_0 = \langle 0, 0 \rangle$$

$$\boxed{r(t) = \langle t, t^2 - t \rangle}$$

Section 6.1

10. Compute $\sum_{i=2}^{5} \frac{i}{i+1}$ $= \underset{i=2}{\frac{2}{2+1}} + \underset{i=3}{\frac{3}{3+1}} + \underset{i=4}{\frac{4}{4+1}} + \underset{i=5}{\frac{5}{5+1}}$

$= \boxed{\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}}$

11. Compute $\sum_{i=1}^{500} (9) =$ $\underbrace{9+9+9+\cdots+9}_{500 \text{ times}} = \boxed{9(500)}$

12. Compute $\sum_{i=3}^{300} (2) =$ $\underbrace{2+2+2+\cdots+2}_{298 \text{ times}} = \boxed{2(298)}$

13. Using the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, find $\sum_{i=1}^{99} 4i$. $= 4 \sum_{i=1}^{99} i$

$= \boxed{4\left(\frac{99(100)}{2}\right)}$

what if we wanted $\sum_{i=3}^{99} 4i = \sum_{i=1}^{99} 4i - \sum_{i=1}^{2} 4i$

$= 4\left(\frac{99(100)}{2}\right) - (4+8)$

14. Write in sigma notation:

a.) $\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}+\sqrt{7} \quad = \quad \sum_{i=3}^{7} \sqrt{i} \qquad \text{or} \qquad \sum_{i=1}^{5} \sqrt{i+2}$

b.) $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25} \quad = \quad \sum_{i=1}^{5} \frac{1}{i^2}$

c.) $1-x+x^2-x^3+x^4-x^5+x^6 \quad = \quad \sum_{i=0}^{6} (-1)^i x^i$