WEEK 14 REVIEW – Finance Part 2 and Markov Chains

AMORTIZATION

Example: A credit card charges 24% annual interest compounded monthly on the unpaid balance. You owe $4000 on this credit card. To pay it off, you stop using it and make monthly payments of $100.
(a) How long until the card is paid off?
(b) How much was paid in interest charges?
(c) How much of the first payment was interest?
(d) How much of the second payment was interest?
(e) How much is still owed on the credit card after two years of making payments?

N= ?] careful $\xrightarrow{\text{solve}}$ 81.3 months
I= 24 $\Longrightarrow$ $\boxed{82 \text{ months}}$
PV= -4000] a sign change in the $
PMT= 100 this is what you owe
FV= 0 (paid off)
P/Y= 12

100$/mo * 82 mo = $8200 sent to C.C.
− 4000
$4200 in interest charges

24%/year * 1 year/12 mo = 2%/mo

1st mo: 4000 * 0.02 = $80 in interest
⟹ 100 − 80 = $20 to pay off balance
still owe 4000 − 20 = $3980
2nd mo: 3980 * 0.02 = $79.60
still owe 3980 − 79.60 =

e) after 2 years we have made 24 payment
82 − 24 = 58 PMT to go

N= 58
I= 24
PV= ? $\rightarrow$ $3414.52
PMT= -100
FV= 0

finance 180,000 − 20,000 = 160000

Example: A house costs \$180,000. You make a down payment of \$20,000 and finance the remainder for 20 years at 5.1% annual interest compounded monthly on the unpaid balance.

(a) How large are the monthly payments?

(b) How much interest is paid in all?

(c) Fill in the blanks on the amortization table below:

a) N = 20 × 12 = 240
I = 5.1
PV = 160000
PMT = ?
FV = 0
P/Y = 12

solve → \$1064.79

$\boxed{5.1\%/\text{yr} \ast \frac{1\text{y}}{12\text{mo}} = 0.425\%/\text{mo} = 0.00425}$

b) $\underbrace{(240)(\$1064.79)}_{\text{Sent the bank}} - \underbrace{160000}_{\text{loan amt}} = \$95,549.60 = $ interest paid

EQ = value of item − what you owe the bank.

ⓐ 160000 * 0.00425 = 680
ⓑ 1064.79 − 680 = 384.79
ⓒ 160000 − 384.79 = 159,615.21
ⓓ 180000 − 159615.21

end of period	remaining payments	payment	interest paid	towards principal	outstanding balance	equity
0	N=240 360	n/a	n/a	n/a	160000	20000
1	239	1064.79	ⓐ 680	ⓑ 384.79	ⓒ 159615.21	ⓓ 20,384.79
2	238	1064.79 −	678.36 =	386.43	159,228.78	20,771.22
FV=0		PMT			solve for PV	180,000 − PV
12	228=N	1064.79			155273.37	24726.63
	240−60				SOLVE↓	180000−PV
60	180=N	1064.79			133,765	46235
					SOLVE↓	
120	120	1064.79			99929	80071
					SOLVE↓	EQ
180	60	1064.79			562,87	123,713

MARKOV CHAINS

① finite stages
② next stage depends only on current stage

Example

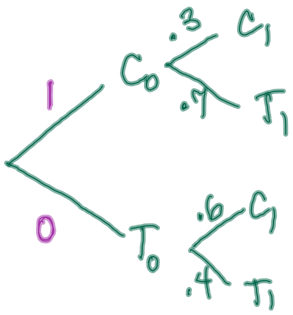
Bob buys a cup of coffee or tea every day. If he buys a cup of coffee, there is a 30% chance he will buy a cup of coffee the next day and a 70% chance he will buy a cup of tea. If he buys a cup of tea, there is a ~~80~~ 40% chance that he will buy a cup of tea the next day and a ~~20~~ 60% chance he will buy a cup of coffee.

(a) Is this a Markov process?

(b) Find the transition matrix.

(c) On his first day back from vacation, Bob buys a cup of coffee. What is the probability that he buys a cup of coffee 3 days later?

(d) What are the long term (steady state) probabilities that Bob buys a cup of coffee or tea?



$$T = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} .3 & .6 \\ .7 & .4 \end{bmatrix} \end{array}$$

always square
col. add to 1
no neg #'s

$X_n = T^n X_0$

$$X_3 = \underbrace{\begin{pmatrix} .3 & .6 \\ .7 & .4 \end{pmatrix}}_{[A]} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{[B]} = [A]\wedge 3 * [B] = \begin{bmatrix} .447 \\ .553 \end{bmatrix} \begin{matrix} C \\ T \end{matrix}$$

44.7% chance he buys coffee

d) $X_L = T X_L \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .3 & .6 \\ .7 & .4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ where x = longterm prob of C, y = long term prob of T

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .3x + .6y \\ .7x + .4y \end{pmatrix} \Leftrightarrow \begin{array}{l} x = .3x + .6y \\ y = .7x + .4y \end{array}$$

$$\begin{array}{l} .3x + .6y = x \\ .7x + .4y = y \end{array} \quad (-x,\ -y) \longleftrightarrow \left.\begin{array}{l} -.7x + .6y = 0 \\ .7x - .6y = 0 \end{array}\right\} \text{parametric}$$

$$x + y = 1$$

$$\left[\begin{array}{cc|c} -.7 & .6 & 0 \\ .7 & -.6 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 6/13 \\ 0 & 1 & 7/13 \\ 0 & 0 & 0 \end{array}\right]$$

≈ 46.15% prob of C
≈ 53.85% prob of T

check with $X_L = T^{100} X_0$

Example

In a certain city elections are held every two years for mayor. There are three political parties in this city, A, B and C. If the current mayor is from the A party, there is a 20% chance that the next mayor will be from the A party, 40% from the B party and 40% from the C party. If the current mayor is from the B party, there is a 50% chance the next mayor will be from the A party and a 50% chance that the mayor will be from the C party. If the current mayor is from the C party, there is a 60% chance the next mayor will be from the C party, a 10% chance the next mayor will be from the A party and a 30% chance the next mayor will be from the C party.

(a) Is this a Markov process? Yes

(b) Find the transition matrix.

(c) On the city's first election there is an equal chance that the mayor comes from parties A, B and C. What is the probability that the mayor is from party A in 8 years? $\Rightarrow$ 4 stages

(d) What is the long term (steady state) distribution of the mayor's political party?

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} .2 & .5 & .1 \\ .4 & 0 & .3 \\ .4 & .5 & .6 \end{bmatrix} \end{array} = T \qquad X_0 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}\begin{array}{c} A \\ B \\ C \end{array}$$

$$X_4 = T^4 X_0 \approx \begin{pmatrix} .2205 \\ .2490 \\ .5305 \end{pmatrix}\begin{array}{c} A \\ B \\ C \end{array}$$

22.05% chance the mayor is party A

$$X_L = TX_L \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .2 & .5 & .1 \\ .4 & 0 & .3 \\ .4 & .5 & .6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} .2x + .5y + .1z &= x \quad -x \\ .4x + \cancel{0y} + .3z &= y \quad -y \\ .4x + .5y + .6z &= z \quad -z \end{aligned} \qquad \begin{aligned} -.8x + .5y + .1z &= 0 \\ .4x - y + .3z &= 0 \\ .4x + .5y - .4z &= 0 \\ x + y + z &= 1 \end{aligned} \quad \xrightarrow{\text{rref}}$$

$$\approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & .2212 \\ 0 & 1 & 0 & .2478 \\ 0 & 0 & 1 & .5316 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\approx$ 22.12% party A

$\approx$ 24.78% party B

$\approx$ 53.16% party C

$$X_L = T^{100} X_0$$

Example

There are three brands of cell phones given to employees of a company. Each year the employee can choose phone brand X, phone brand Y or phone brand Z. If an employee has a brand X phone, he will choose a brand X phone again the next year. If an employee has a brand Y phone, there is an equal chance that he will choose brand X, Y or Z the next year. If a person has a brand Z phone, he will choose a brand X phone 50% of the time, a brand Y phone 25% of the time and a brand Z phone 25% of the time.

(a) Is this a Markov process? Yes

(b) Find the transition matrix.

(c) Initially all employees are given brand Z phones. What is the distribution of phone ownership in 2 years?

(d) What is the long term (steady state) distribution of phone ownership? all have brand X

absorbing

$$T = \begin{array}{c c} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{pmatrix} 1 & 1/3 & .5 \\ 0 & 1/3 & .25 \\ 0 & 1/3 & .25 \end{pmatrix} \end{array} \qquad X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{c} X \\ Y \\ Z \end{array}$$

$$X_2 = T^2 X_0 = \begin{bmatrix} 17/24 \\ 7/48 \\ 7/48 \end{bmatrix} \approx$$

70.83% brand X
14.58% brand Y and Z

regular? T^n has only positive entries (no zeros)

Example

Classify the following matrices as a regular transition matrix (not absorbing), not a regular transition matrix (absorbing), or not a transition matrix.

must be square
no neg #'s
sum of col = 1

(a) $\begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix}^{100} \Rightarrow \begin{bmatrix} 2/7 & 2/7 \\ 5/7 & 5/7 \end{bmatrix}$ regular

(b) $\begin{bmatrix} 0.5 & 0.8 \\ 0.5 & 0.6 \end{bmatrix}$ → not a transition matrix

1.4

(c) $\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0 \\ 0.25 & 1 \end{bmatrix}^{100} \end{array} = \begin{bmatrix} \cancel{3 \times 10^{-18}} \; 0 & 0 \\ 1 & 1 \end{bmatrix}$ not regular