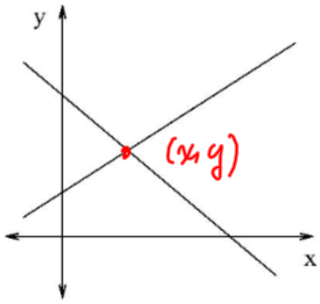
WEEK 2 REVIEW
SYSTEMS OF LINEAR EQUATIONS

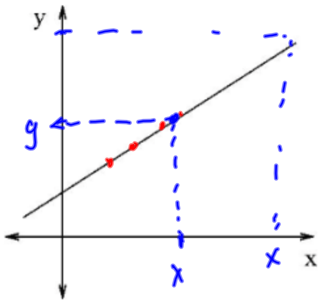
When you have a system of two equations and two unknowns we have three possibilities for the two lines:

The two lines intersect. The solution is the single point of intersection.

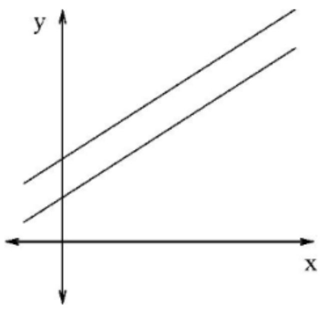


The two lines are the same. The solution is the entire line.

parametric soln



The two lines are parallel. No intersection therefore no solution.



Same slope
different
intercept

$(x,y) = \_\_\_\_$

Example: Solve the following systems of linear equations.

(a) $x + 2y = 12 \Rightarrow x = 12 - 2y \Rightarrow x = 2$

$2x + 3y = 19$

$2(12-2y) + 3y = 19 \rightsquigarrow y = 5$

$(2,5)$

(b) $2x - 4y = 8 \Rightarrow -4y = 8 - 2x \Rightarrow y = -2 + x/2$

$-x + 2y = 4$

$-x + 2(-2 + x/2) = 4 \Rightarrow$

$\cancel{-x} - 4 + \cancel{x} = 4 \qquad -4 = 4 \Rightarrow$ FALSE NO SOLN

slope-int form $y = x/2 - 2$ and other is $y = x/2 + 2$

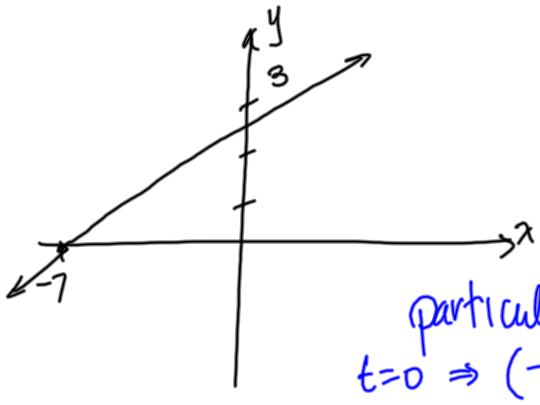
(c) $-x + 3y = 7 \Rightarrow x = 3y - 7$

$2x - 6y = -14$

$2(3y-7) - 6y = -14$ TRUE

$6y - 14 - 6y = -14 \Rightarrow 0 = 0$

$\rightsquigarrow y = (1/3)x + 7/3$ (both same)



$(x,y) = (3y-7, y)$

or $(3t-7, t)$

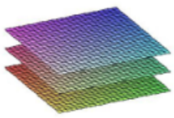
t is any real #
$t \in \mathbb{R}$

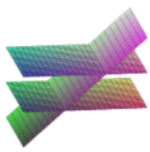
particular soln?
$t=0 \Rightarrow (-7,0)$
$t=1 \Rightarrow (-4,1)$ etc

Number of Solutions Theorem

If the number of equations in a system of linear equations is equal to or greater than the number of variables, the system may have

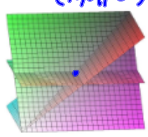
- No solution



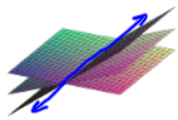


- Exactly one solution

(x,y,z)



- A parametric solution



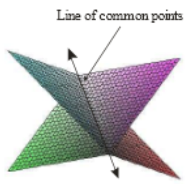
1 parameter

2 parameters

If the number of equations in a system of linear equations is less than the number of variables, then the system may have

- No solution

- A parametric solution

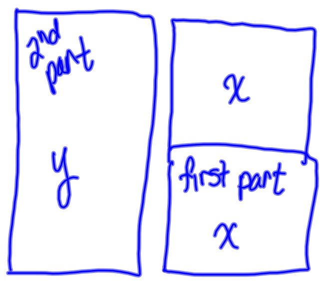


One solution is NOT POSSIBLE

WORD PROBLEMS (set up only)

Example: Jane invests \$10,000 in three ways. With one part she buys mutual funds with a return of 6.5% per year. The second part (which is twice as large as the 1st part) is used to buy government bonds that pay 6% per year. The rest is put into a savings account paying 5% per year. In the 1st year her average return was 6.05%.. How much did she invest in each way?

x = amt of \$ invested in M Funds
y = " " " " " G Bonds
z = " " " " " Savings Acct



$$x + y + z = 10{,}000 \quad \text{(total \$ invested)}$$

$$y = 2x \quad \text{(ratio of MF to GB)}$$

$$0.065x + 0.06y + 0.05z = 0.0605(10000) = 605 \quad \text{(int earned)}$$

Example: Farmer Blue has 100 acres available to plant white and yellow corn. Each acre of white corn will yield 95 bushels of corn and each acre of yellow corn will yield 120 bushels of corn. He wants to have three times as many bushels of white corn than he does of yellow corn. How many acres of each type of corn should Farmer Blue plant?

x = # of acres of white corn

y = " " " yellow corn

$x + y = 100$ (total acres)

$95x = 3(120y)$ (ratio)

GAUSS-JORDAN

Example
Solve the following system of linear equations:

"messy"

$$-3x - 6y + 6z = -3$$
$$2x + 7y + 2z = -1$$
$$-x - 6y - 7z = 3$$

if we had "nice" $x = a$, $y = b$, $z = c$ then $(x,y,z) = (a,b,c)$

1. Any two equations may be interchanged.
2. An equation may be multiplied by a non-zero constant.
3. A multiple of one equation may be added to another equation.

Augmented Matrix
A matrix is in Reduced-Row Echelon Form if
1. Each row consisting entirely of zeros lies below any row having non-zero entries.
2. The 1st non-zero entry in any row is a 1 (called a leading 1)
3. In any two successive (non-zero) rows the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column contains a leading 1, the rest of the column is 0. ✱

IMPORTANT!

- Only consider entries to the LEFT of the vertical line when Appling the definition of RREF.
- If a matrix is in RREF form, it may have one solution, no solution or a parametric solution.
- When you PIVOT on an element, you make the element a 1 (by multiplication) and the rest of the column 0 (by addition)

$$\begin{array}{cccc} x & y & z & = \# \\ \end{array}
\left[\begin{array}{ccc|c} -3 & -6 & 6 & -3 \\ 2 & 7 & 2 & -1 \\ -1 & -6 & -7 & 3 \end{array}\right]
\rightsquigarrow
\begin{array}{cccc} x & y & z & = \# \\ \end{array}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]
\quad \begin{array}{l} x = a \\ y = b \\ z = c \end{array}$$

Example: Solve the following system of linear equations using Gauss Jordan and showing all your work

$$-3x-6y+6z=-3$$
$$2x+7y+2z=-1$$
$$-x-6y-7z=3$$

$$\to \left[\begin{array}{ccc|c} -3 & -6 & 6 & -3 \\ 2 & 7 & 2 & -1 \\ -1 & -6 & -7 & 3 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1 \to R_1}$$

① Pivot on Row 1, Col 1 (make this a 1 and the rest of the col zeros)

ⓐ make this element a 1

ⓑ make the zeros

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 7 & 2 & -1 \\ -1 & -6 & -7 & 3 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 6 & -3 \\ 0 & -4 & -9 & 4 \end{array}\right]$$

② Pivot on Row 2 Col 2

ⓐ make this element a 1

ⓑ rest of coln 2 zeros

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -4 & -9 & 4 \end{array}\right] \xrightarrow[4R_2+R_3 \to R_3]{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

③ Pivot on R3 C3

ⓐ make this a 1

$$\xrightarrow{-1\cdot R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-2R_3+R_2 \to R_2]{6R_3+R_1 \to R_1} \begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$x=3$
$y=-1$
$z=0$

Example: Solve the following systems of linear equations:

(a)
$$\begin{aligned} 2x+y-z&=0\\ 3x-y+2z&=1\\ x-2y+3z&=2 \end{aligned}$$

(b)
$$\begin{aligned} 2x+y-4z&=10\\ x+2y+z&=5\\ x+y-z&=5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0\\ 3 & -1 & 2 & 1\\ 1 & -2 & 3 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cccc} x & y & z & = \# \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & .2 & 0\\ 0 & 1 & -1.4 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$x + .2z = 0$

$y - 1.4z = 0$

$0 = 1$

NO SOLN

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & 10\\ 1 & 2 & 1 & 5\\ 1 & 1 & -1 & 5 \end{array}\right] \xrightarrow{\text{ref}} \begin{array}{c} \begin{array}{cccc} x & y & z=t & \# \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & -3 & 5\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x - 3\cancel{z}^{\,t} = 5$

$y + 2\cancel{z}^{\,t} = 0$

$0 = 0$

$x - 3t = 5$

$y + 2t = 0$

$z = t$

$x = 3t + 5$

$y = -2t$

$\boxed{(x, y, z) = (3t+5, -2t, t)}$

t any $\mathbb{R}$

part soln $t=0$: $(5,0,0)$ $t=1$: $(8,-2,1)$

$(5,2,0)$ is NOT A SOLN

Example: Are the matrices below in Reduced-Row Echelon Form? If yes, write the solution to the system of equations. If not, complete the next BEST step in Gauss-Jordan.

NOT RREF

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 0 & e \\ 0 & 1 & 2 & f \\ 0 & 0 & 3 & 4 \end{array}\right] \xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & e+2f \\ 0 & 1 & 2 & f \\ 0 & 0 & 3 & 4 \end{array}\right]$$

IS IN RREF form

$$\begin{array}{cccc} x & y & z & = \# \\ & t & & \end{array}$$

$$B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = 4$
$z = 1$
$0 = 0$
$y = t$

$(x, y, z) = (4, t, 1)$

t any $\mathbb{R}$

$$\begin{array}{cccc} x & y & z & = \# \end{array}$$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$x + z = 0$
$y + 2z = 0$
$\boxed{0 = 2}$

NO SOLN

Example: A zoo is looking to acquire some lions, tigers and bears. The zoo has 2800 square feet of space available and \$850 for transportation costs. A lion needs 200 square feet of space and costs \$50 to transport. A tiger needs 400 square feet of space and costs \$150 to transport. A bear needs 400 square feet of space and costs \$50 to transport. How many lions, tigers and bears can the zoo get?

x = # of lions, y = # of tigers, z = # of bears

$200x + 400y + 400z = 2800$ (sq ft)
$50x + 150y + 50z = 850$ (\$ trans)

$$\left[\begin{array}{ccc|c} 200 & 400 & 400 & 2800 \\ 50 & 150 & 50 & 850 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{cccc} x & y & z \to t & \\ \left[\begin{array}{ccc|c} 1 & 0 & 4 & 8 \\ 0 & 1 & -1 & 3 \end{array}\right] \end{array}$$

$x + 4\cancel{z}^{\,t} = 8 \Rightarrow x = 8 - 4t$
$y - \cancel{z}_{\,t} = 3 \Rightarrow y = t + 3$
$z = t$

$(x, y, z) = (8 - 4t, t + 3, t)$ t = # of bears

$t = 0$: $(8, 3, 0)$ 8 lions, 3 tigers, and 0 bears or
$t = 1$: $(4, 4, 1)$ 4 lions, 4 tigers, and 1 bear or
$t = 2$: $(0, 5, 2)$ 0 " 5 " " 2

MATRICES

Matrices are a compact way of showing a large amount of data. You must be able to put data into a matrix and then add, subtract and multiply matrices.

It is important to label the rows and columns of your matrix if it is a word problem. The size of a matrix is given as

number of rows x number of columns.

If we are given the matrix $A = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{pmatrix} 1 & -2 & a \\ 0 & b & 3 \end{pmatrix} \\ \begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \end{array}$

It is A is a 2x3matrix as it has 2 rows and 3 columns.

The elements of a matrix are described in terms of its row number and column number (in that order). Element a_{ij} is in row i and column j.

DATA MATRICES

A chain of shoe stores has two stores in the area, Store A and Store B. Each carries children's, women's and men's shoes. In one week Store A sold 150 pairs of children's shoes, 90 pairs of women's shoes and 40 pairs of men's shoes. Store B sold 50 pairs of children's shoes, 120 pairs of women's shoes and 60 pairs of men's shoes. Show this data in a 2x3matrix.

$$\begin{array}{cc} & \begin{array}{ccc} C & W & M \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 150 & 90 & 40 \\ 50 & 120 & 60 \end{pmatrix} \end{array}$$

MATRIX "ALGEBRA"

Two matrices are equal if each corresponding pair of elements in the matrices are equal. MUST BE THE SAME SIZE

Two matrices are added by adding the corresponding pairs of elements. MUST BE THE SAME SIZE

A matrix is multiplied by a constant by multiplying each element of the matrix by the constant.

Example: Find the values of w, x, y, and z below.

$$2\begin{pmatrix} w & -1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -2 & x \\ y & 1 \end{pmatrix}^T = \begin{pmatrix} 5 & 0 \\ 6 & z \end{pmatrix}$$

$$\begin{pmatrix} 2w & -2 \\ 8 & 6 \end{pmatrix} + \begin{pmatrix} -2 & y \\ x & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 6 & z \end{pmatrix}$$

$$\begin{pmatrix} 2w-2 & -2+y \\ 8+x & 6+1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 6 & z \end{pmatrix}$$

$2w-2=5 \to w=3.5$

$8+x=6 \to x=-2$

$-2+y=0 \to y=2$

$6+1=z \to z=7$

A matrix is transposed when the rows and columns of the matrix are swapped. $(a_{ij})^T \to a_{ji}$

$$\begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{pmatrix} 2 & 3 & 4 \\ 0 & 7 & 6 \end{pmatrix}^T \\ \begin{array}{ccc} a_{21} & a_{22} & \phantom{a_{23}} \end{array} \end{array} = \begin{pmatrix} 2 & 0 \\ 3 & 7 \\ 4 & 6 \end{pmatrix}$$

2×3 $\qquad$ 3×2