CHAPTER 13 – FAIR DIVISION

A fair division procedure is ***equitable*** if each player believes he or she received the same fractional part of the total value.

A fair division procedure is ***envy-free*** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.

A fair division procedure is said to be ***Pareto-optimal*** if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

Adjusted Winner Procedure:

Step 1. Each party distributes 100 points over the items in a way that reflects their relative worth to that party.

Step 2. Each item is initially given to the party that assigns it more points. If there is a tie, the item is not assigned.

Step 3. Each party totals up the number of points it has received and the party that has received the fewest number of points is now given the item that that had a tie.

Step 4. If the number of points each party has is tied, the procedure is complete. If one party has more points, it is named party A and the party with fewer points in named party B.

Step 5. Items are now transferred from party A to party B until the point totals are equal. Fractional transfers are allowed. Transfers are determined using ***point ratios***. Transfer item with lowest ratio.

To determine an item's ***point ratio***, find the fraction

$$\frac{\text{A's point value of the item}}{\text{B's point value of the item}}$$

This procedure is well suited to dividing several items between two people or parties.

Example 1

Rand and Mat will split 4 items using the adjusted winner procedure with the point values listed below. How are the items distributed?

Rand gets the coin and the bag

Item	Rand (50)	Mat (75)	Point ratio
Gold coin	25	5	—
Saddle Bag	25	20	—
Cape	25	35	$35/25 = 1.4$
Hat	25	40	$40/25 = 1.6$

→ Share this one with lowest ratio (keep x and give 1−x)

Rand's points = Mat's points

$$\underbrace{25}_{\text{coin}} + \underbrace{25}_{\text{bag}} + \underbrace{25(1-x)}_{\text{cape}} = \underbrace{40}_{\text{hat}} + \underbrace{35(x)}_{\text{cape}}$$

$$25+25+25-25x = 40+35x$$

$$75-25x = 40+35x$$

$$+25x \qquad +25x$$

$$75 = 40+60x$$

$$-40 \qquad -40$$

$$\frac{35}{60} = \frac{60x}{60} \Rightarrow x = 35/60 = 7/12 \quad (0.5833)$$

Rand has the coin, the bag and $1 - 7/12 = 5/12^{th}$ of the cap

Mat has the hat and $7/12^{th}$ of the cap

Rand's points: $25 + 25 + 25 \cdot 5/12 \approx 60.4167$

Mat's points: $40 + 35(7/12) \approx 60.4167$

Example 2

Katniss and Peeta will split 5 items using the adjusted winner procedure using the point values listed below. How are the items distributed?

Peeta has the Knife and the food

60 +20 = 80   75

ratio

Item	Katniss	Peeta
Bow and Arrows	50	0
Water bottle	20	20
Knife	15	35
Food	5	40
Blanket	10	5

$50/0 = $ BIG #

$20/20 = 1 \Rightarrow$ share this item

$10/5 = 2$

Katniss gets x

Peeta gets $1-x$

Katniss's pts = Peeta's points

$$\underbrace{\underset{B\&A}{50} + \underset{\text{blanket}}{10}}_{60} + \underset{WB}{20(x)} = \underbrace{\underset{K}{35} + \underset{F}{40}}_{75} + \underset{WB}{20(1-x)}$$

$$60 + 20x = 75 + 20 - 20x = 95 - 20x$$

$$-60 + 20x \qquad -60 + 20x$$

$$40x = 35 \Rightarrow x = 35/40 = 7/8 = 0.875$$

Katniss has the B&A, the blanket 7/8th of the water bottle ($\Rightarrow$ each have 77.5 pts)

Peeta has Knife, the food and 1/8th of the water bottle

Example 3

Ozma and Dorothy will split some jewelry using the adjusted winner procedure using the point values listed below. How are the items distributed?

Ozma has #1, #6 and #7

44 +11 = 55   84 - 14 = 70   ratios

	Item	Ozma	Dorothy	
#1	Gold Crown	10	5	
#2	Silver Crown	10	20	20/10 = 2 ⇒ transfer this 2nd ⇒ D keeps x, Oz gets 1-x
#3	Diamond Bracelet	15	20	20/15 = 1.333
#4	Sapphire Bracelet	11	14	14/11 = 1.273 ⇒ transfer this first ⇒ Dorothy keeps x, Ozma gets 1-x
#5	Emerald Bracelet	20	30	30/20 = 1.5
#6	Ruby Bracelet	22	8	
#7	Gold Earrings	12	3	

OZMA'S = DOROTHY

$$\underset{\#1}{10} + \underset{\#6}{22} + \underset{\#7}{12} + \underset{\#4}{11(1-x)} = \underset{\#2}{20} + \underset{\#3}{20} + \underset{\#5}{30} + \underset{\#4}{14x}$$

$$55 - 11x = 70 + 14x$$

NEG??? $\Rightarrow -15 = 25x$ → transfer entire item

$$\underset{\#1}{10} + \underset{\#4}{11} + \underset{\#6}{22} + \underset{\#7}{12} + \underset{\#3}{15(1-x)} = \underset{\#2}{20} + \underset{\#5}{30} + \underset{\#3}{20(x)}$$

$$70 - 15x = 50 + 20x$$

$$+15x \qquad +15x$$

$$70 = 50 + 35x$$

$$-50 \qquad -50$$

$$20 = 35x \Rightarrow x = 20/35 = 4/7$$

Ozma has #1, #4, #6, #7 and $(1-4/7) = 3/7$ of item #3

D has #2, #5 and 4/7 of #3

The Knaster Inheritance Procedure

Step 1. The heirs – independently and simultaneously – submit monetary bids for the object.

Step 2. The high bidder is awarded the object and he or she places all but $1/n$ of his or her bid in a kitty.

Step 3. Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.

Step 4. The remaining money in the kitty is divided equally

This procedure is well suited to dividing a few items between two people or more people.

$90-\frac{1}{3}(90)=60$

Example 4

Janice, Cindy and Teri receive a coat. To decide who gets the coat they use the Knaster Inheritance Procedure. Janice bids \$90, Cindy bids \$75 and Teri bids \$60. What are the results of the division?

(2) Janice gets the coat. She puts $\frac{2}{3}\times 90=\$60$ in the kitty

(3) Cindy takes $\frac{1}{3}\cdot 75=\$25$ from the kitty

Teri takes $\frac{1}{3}\cdot 60=\$20$ from the kitty

(4) $\underset{\text{kitty}}{60}-\underset{\text{cindy}}{25}-\underset{\text{Teri}}{20}=\$15$ left $\Rightarrow \frac{\$15}{3}=\$5$ each

$\Rightarrow$ Cindy gets \$25 + \$5 = \$30, Teri gets \$20 + \$5 = \$25

Janice gets the coat and pays $-60+5 \Rightarrow \$55$

Example 5

John, Paul, George, and Ringo receive a piano and a drum set . To decide who gets these items they use the Knaster Inheritance Procedure.
(1) John bids \$800 on the piano and \$500 on the drums,
Paul bids \$720 on the piano and \$440 on the drums,
George bids \$600 on the piano and \$620 on the drums.
Ringo bids \$400 on the piano and \$400 on the drums.
What are the results of the division?

PIANO

(2) John gets the piano and puts $3/4 \cdot 800 = \underline{\underline{600}} = 800 - 1/4 \cdot 800$ in kitty

(3) Paul takes $720/4 = \$180$ from kitty
G takes $600/4 = \$150$ " "
R takes $400/4 = \$100$ " "

$600 - 180 - 150 - 100 = 170$ left
split ⇒ $170/4 = 42.50$ back each

DRUMS

(2) George gets the drums and puts $3/4(620) = 465 = 620 - 1/4(620)$ in kitty
John takes $500/4 = 125$
Paul takes $440/4 = 110$
Ringo takes $400/4 = 100$

$465 - 125 - 110 - 100 = \$130$ left
$\$130/4 = \$32.50$ each

John : has piano pays $-600 + 42.50 + 125 + 32.50 = \$-400$
Paul : $180 + 42.50 + 110 + 32.50 = \$365$
George : has drums pays $-465 + 150 + 42.50 + 32.50 = -\$240$
Ringo : $100 + 42.50 + 100 + 32.50 = \$275$

Divide and Choose Procedures

When an item is to be divided between two players, one player will divide the item into two pieces that in the dividers opinion are of equal value.

We will divide a "cake" by making a single vertical cut in the cake to create two pieces.

The chooser will pick the left or right piece to get, in the chooser's opinion, at least half of the cake.

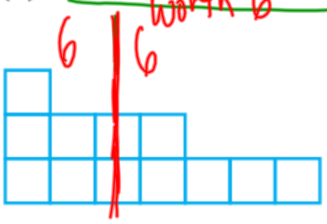
<u>*Example 7*</u>

12 units

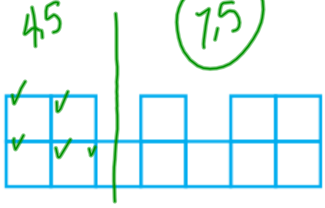
Robert and Bob will divide a cake using divide and choose.

(a) With Robert dividing, which side will Bob choose?

both sides are worth 6



Robert's view

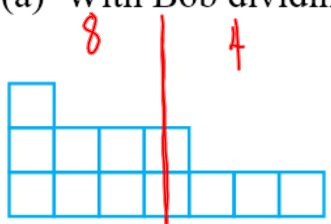


Bob's view

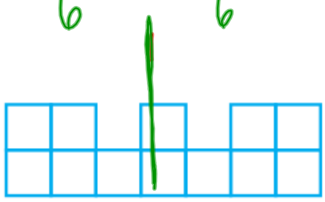
Robert gets left side worth 6 units

Bob picks the right side and gets 7.5 units

(a) With Bob dividing, which side will Robert choose?



Robert's view



Bob's view

Robert chooses left side and gets 8 units
and Bob gets right side for 6 units

A ***cake-division procedure*** for n players is a procedure that the players can use to allocate a cake among them so that each player has a strategy that will guarantee that player a piece with which he or she is "satisfied."

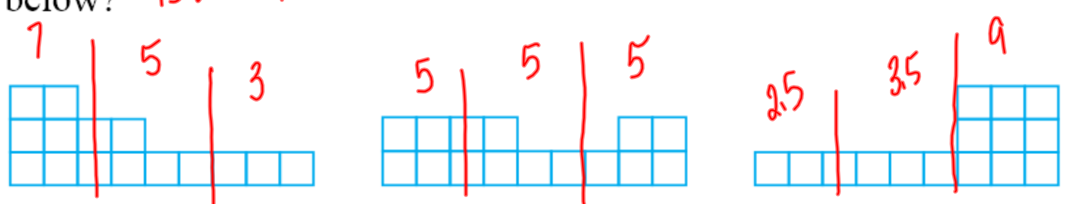
A cake-division procedure for n player is called ***proportional*** if each player's strategy guarantees that player a piece that is worth at least $1/n$ of the whole, in that player's estimation.

The Steinhaus Proportional Procedure (Lone Divider) for Three Players

Step 1. The players (A, B, and C) let player A be the divider.
Step 2. Player A divides the cake into three equal pieces, i, ii, and iii
Step 3. If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
Step 4. If players B and C both want the same piece, they give a not wanted piece to player A. The remaining two pieces are combined and then B divides and C chooses.

<u>*Example 8*</u>

William, Bill and Jane will divide a cake using the Steinhaus proportional procedure. The divider will be Bill ~~and~~ William and Jane will choose. How will the cake be divided if the view of each person is as shown below? 15 units of cake



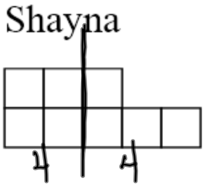
William's view Bill's view Jane's view

W takes left for 7, Jane takes right for 9 and Bill gets center for 5

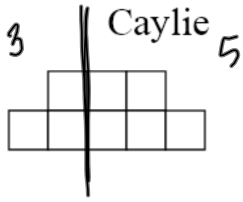
SAMPLE EXAM QUESTIONS FROM CHAPTER 13

Suppose that Shayna and Caylie view a cake as shown below. They agree to divide the cake using the divide-and-choose procedure. 8 units

Shayna



Caylie



1. If Shayna divides the cake, where will the cut be made?

A) 3 columns from the left B) 2 ½ columns from the left

C) 2 columns from the left D) 1 ½ columns from the left

2. If Shayna divides the cake, which side will Caylie choose?

A) The right side B) The left side

5. Match the terms with the correct definition:

(a) Equitable ____________ (b) Pareto-optimal i

(i) A fair-division procedure that produces an allocation such that no other allocation can make one person better off without making another person worse off. P-O

(ii) A fair-division procedure that can guarantee that each person has a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player. E-F

(iii) A fair-division procedure that results in a division that each person believes he or she received the same fractional part of the total value as the other people. eq

(iv) ~~A fair-division process where people take turns choosing items.~~

(v) ~~A fair-division process where one person divides and the other person chooses.~~

6. Lucy and Sandy must make a fair division of a printer, a microwave and a lamp. They place point values on the objects as shown below. Using the adjusted winner procedure, what do Lucy and Sandy receive?

Object	Lucy's points (90)	Sandy's points (50)
Printer	40	30
Microwave	10	50
Lamp	50	20

ratio

$40/30 = 1.333 \Rightarrow$ share this
x for Lucy
$1-x$ for Sandy

$50/20 = 2.5$

Lucy Sandy

$$\underbrace{40x}_{P} + \underbrace{50}_{L} = \underbrace{50}_{M} + \underbrace{30(1-x)}_{L} \Rightarrow$$

$40x+50 = 80 - 30x$

$70x = 30 \Rightarrow x = 3/7$

Lucy gets the lamp and 3/7 of the printer.
Sandy gets the micro and 4/7 of the printer

7. Nancy, Elayne, and Teri must make a fair division of a boat left to them by their father using the Knaster inheritance procedure. The values they bid on the boat are Nancy - \$4200, Elayne - \$3600, and Teri - \$3000. What are the results of the division?

① ② Nancy gets the boat. Puts $4200 \cdot 2/3 = 2800 = 4200 - 1/3 \cdot 4200$

③ Elayne takes $3600/3 =$ \$1200 from kitty
Teri takes $3000/3 =$ \$1000 from kitty

$2800 - 1200 - 1000 = 600$

$\frac{600}{3} = 200$

Nancy gets the boat. Pays $-2800 + 200 = -\$2600$
Elayne gets \$1200 + 200 = \$1400
Teri gets \$1000 + 200 = \$1200