Setting Up Linear Programming Problems

Example

A company produces handmade skillets in two sizes, big and giant. To produce one big skillet requires 3 lbs of iron and 6 minutes of labor. To produce one giant skillet requires 4 lbs of iron and 3 minutes of labor. The profit for each big skillet is $20 and the profit for each giant skillet is $25. If 1000 lbs of iron and 20 hours of labor are available each day, how many of each size skillet should be made to maximize profits?

$x = $ # of big skillets

$y = $ # of giant skillets

$P = $ profit in $

OBJ : max $P = 20x + 25y$

SUBJECT TO

$3x + 4y \leq 1000$ (lb of iron)

$6x + 3y \leq 1200$ (min of labor)

$x \geq 0, y \geq 0$ (non-neg)

Example

Annika has $3000 available to invest, with which she buys shares of stock in the Bass Company and Forte Inc. A share of Bass stock costs $10 and pays dividends of $3.00 per share each year. Forte stock costs $50 per share and pays dividends of $4.00 per share each year. Annika wishes to purchase no more than 80 total shares of stock in order to avoid paying higher trading fees to her broker. Due to the volatility of the Bass stock, her broker recommends that she invest in at most 50 shares of that company's stock. Determine the number of shares Annika should buy of each kind of stock in order to maximize the amount from dividends that she will receive at the end of the first year.

$x = $ # of sh of Bass

$y = $ # of sh of forte

$D = $ div. in $ from stocks

Max $D = 3x + 4y$

SUB TO

$x + y \leq 80$ (total shares)

$10x + 30y \leq 3000$ (total $)

$x \leq 50$ (limit on Bass)

$x \geq 0, \quad y \geq 0$

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Example

A dietitian is to prepare two foods in order to meet certain requirements. Each ounce of food I contains 100 units of vitamin C, 40 units of vitamin D and 20 units of vitamin E and costs 25 cents. Each ounce of food II contains 10 units of vitamin C, 80 units of vitamin D and 15 units of vitamin E and costs 15 cents. The mixture of the two foods is to contain at least 260 units of vitamin C, 320 units of vitamin D and 120 units of vitamin E. How many ounces of each type of food should be used in order to minimize the cost?

x = # of oz of food I
y = # of oz of food II
C = cost in \$.

OBJ MIN $C = 0.25x + 0.15y$

SUB TO

$100x + 10y \geq 260$ units of vit C

$40x + 80y \geq 320$ units of vit D

$20x + 15y \geq 120$ units of vit E

$x \geq 0, \ y \geq 0$

Example

A craftsman has 150 units of wood, 90 units of glue and 150 units of paint. A small picture frame requires 1 unit of wood, 1 unit of glue and 2 units of paint while a large picture frame requires 5, 2 and 1 respectively. If a small frame sells for $175 and a large frame for $400, how many of each should be made to maximize the revenue?

x = # of sm. frames
y = # of lg frames
R = Revenue in $

max $R = 175x + 400y$

SUB TO

$1x + 5y \leq 150$ units of wood
$1x + 2y \leq 90$ units of glue
$2x + 1y \leq 150$ units of paint
$x, y \geq 0$

Graphing Systems of Linear Inequalities

The general forms for linear inequalities are

$$ax + by + c \geq 0$$
$$ax + by + c > 0$$
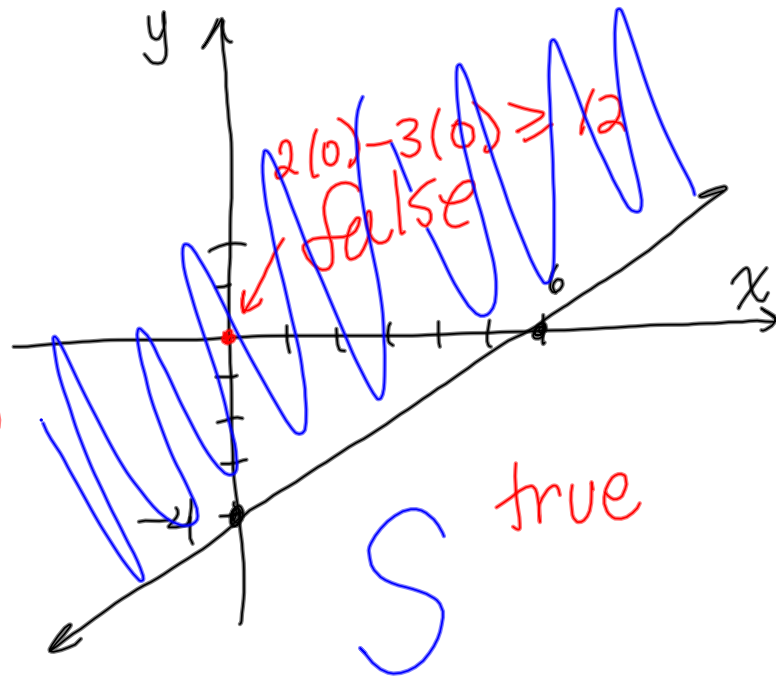$$ax + by + c \leq 0$$
$$ax + by + c < 0$$

Example
Graph $2x - 3y \geq 12$

$2x - 3y = 12$
$(0, -4)$
$(6, 0)$

(handwritten on graph) $2(0) - 3(0) \geq 12$ false ; true ; S

NOTE - if your line passes through the origin, you must take a different point for a test point.
If our inequality had $\geq$ or $\leq$ we draw the bounding line as a solid.

If our inequality had $>$ or $<$ we draw the bounding line as DASHED .

The region that satisfies our inequality is called the *feasible region*. This is the region that is white (unshaded). Please label it with an *S*.

What if we have two inequalities (a system)? The feasible region (solution) will be where they are both true at the same time.
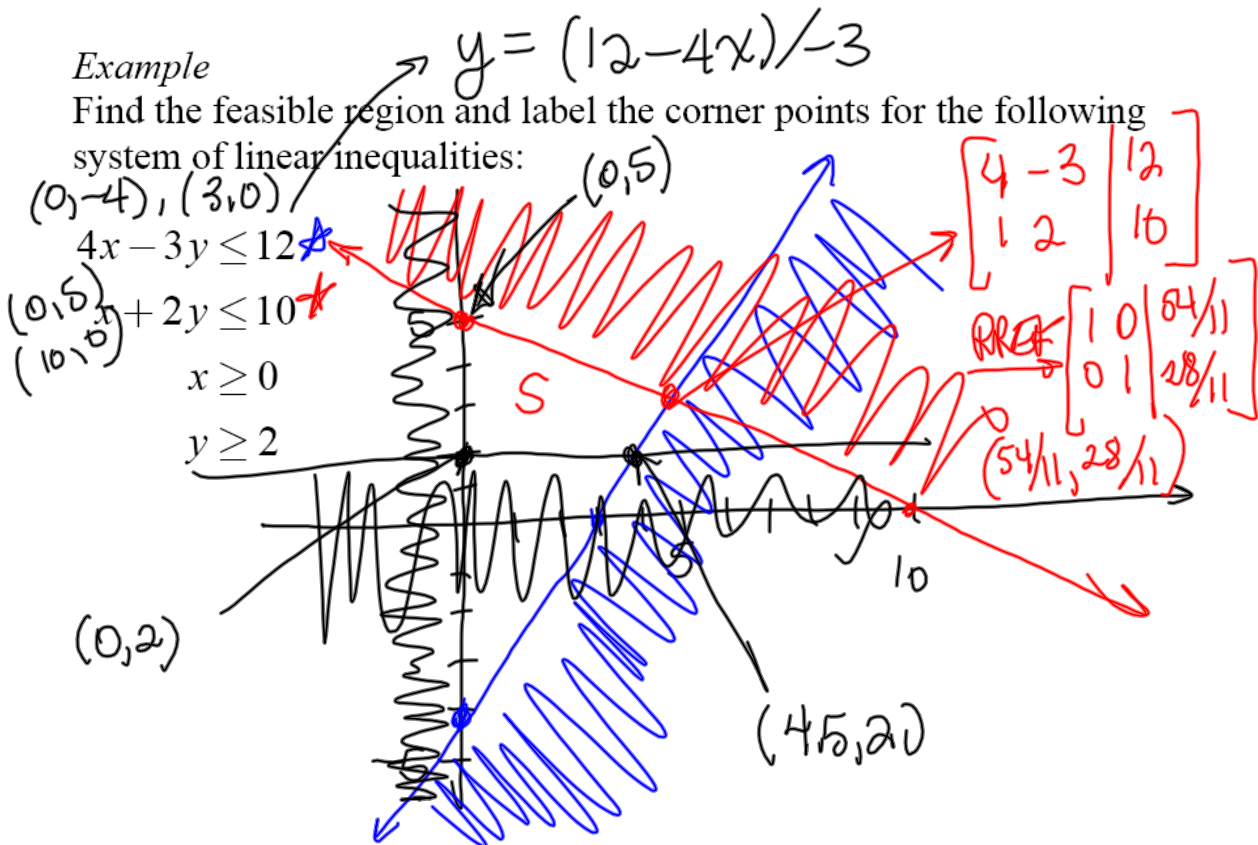
Example

Find the feasible region for the system

$$(0,3), (2,0)$$
$$3x + 2y \geq 6$$
$$x \leq 4 \quad \text{vert line @ } x=4$$

S

$(4,-3)$
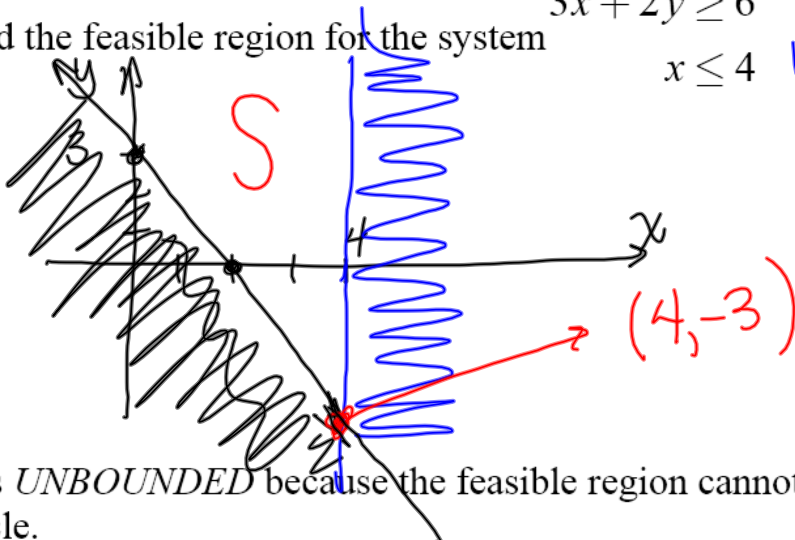
It is *UNBOUNDED* because the feasible region cannot be enclosed in a circle.

If S can be enclosed by a circle, it is called *BOUNDED*.

Example

$$y = (12 - 4x)/-3$$

Find the feasible region and label the corner points for the following system of linear inequalities:

$(0,-4), (3,0)$
$$4x - 3y \leq 12$$
$(0,5)$
$(10,0)$ $x + 2y \leq 10$
$$x \geq 0$$
$$y \geq 2$$

$(0,5)$

S

$\begin{bmatrix} 4 & -3 & | & 12 \\ 1 & 2 & | & 10 \end{bmatrix}$

RREF $\begin{bmatrix} 1 & 0 & | & 54/11 \\ 0 & 1 & | & 28/11 \end{bmatrix}$

$(54/11, 28/11)$

$(0,2)$

10

$(4.5, 2)$

Principles of Linear Programming

Annika has $3000 available to invest, with which she buys shares of stock in the Bass Company and Forte Inc. (…) Determine the number of shares Annika should buy of each kind of stock in order to maximize the amount from dividends that she will receive at the end of the first year.

Let x = the number of shares of Bass stock
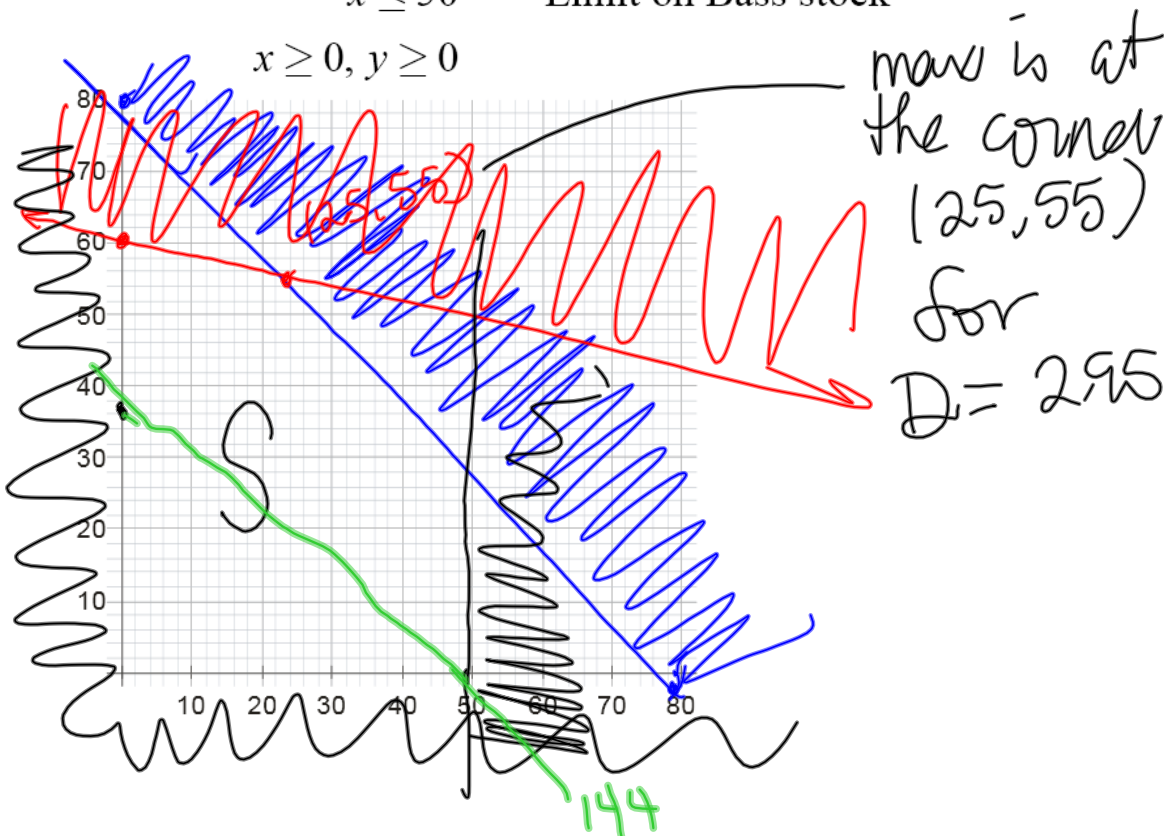Let y = the number of shares of Forte stock
Let D = the total amount of dividends earned

OBJECTIVE: Maximize $D = 3x + 4y$

SUBJECT TO:

$$\begin{bmatrix} 1 & 1 & 80 \\ 10 & 50 & 3000 \end{bmatrix}$$

$x + y \le 80$ Total number of shares $(0, 80), (80, 0)$

$10x + 50y \le 3000$ Total $ invested $(0, 60)$ and $(300, 0)$

$x \le 50$ Limit on Bass stock

$x \ge 0, y \ge 0$

max is at the corner $(25, 55)$ for $D = 295$

Can she buy 60 shares of Bass and 10 shares of Forte?

No $\longrightarrow$ too much Bass

Can she buy 10 shares of Bass and 65 shares of Forte?

$10(10) + 50(65) = 3350$ too much $

Can she buy 40 shares of Bass and 50 shares of Forte?

too many Shares

Can she buy 30 shares of Bass and 40 shares of Forte? Yes

$D = 3(30) + 4(40) = \$250$

If Annika buys 48 shares of Bass and 0 shares of Forte, how much does she earn in dividends?

$D = 3(48) + 4(0) = \$144$

If Annika buys 0 shares of Bass and 36 shares of Forte, how much does she earn in dividends?

$D = 3(0) + 4(36) = \$144$

$D = \boxed{3x + 4y = 144}$

$y = -3/4\, x + 36$

$D = \boxed{192 = 3x + 4y}$

$y = -3/4\, x + 48$

Annika should buy 25 shares of Bass and 55 shares of Forte to earn the max div of $295.

Solving Linear Programming Problems

Every linear programming problem has a feasible region associated with the constraints of the problem. These feasible regions may be bounded, unbounded or the empty set.

To find the solution (that is, where the maximum or minimum value occurs), we will use the two theorems below.

Theorem 1   If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set S, associated with the problem. Furthermore, if the objective function P is optimized at two adjacent vertices of S, then it is optimized at every point on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Theorem 2   Suppose we are given a linear programming problem with a feasible set S and an objective function $P = ax + by$.
- **Case 1** If S is bounded, then P has both a maximum and a minimum value on S.
- **Case 2** If S is unbounded and both a and b are nonnegative, then P has a minimum value on S provided that the constraints defining S include the inequalities $x \geq 0$ and $y \geq 0$.
- **Case 3** If S is the empty set, then the linear programming problem has no solution; that is, P has neither a maximum nor a minimum value.

Example

A company produces handmade skillets in two sizes, big and giant. (…)
how many each size skillet should be made to maximize profits?

x = the number of big skillets produced
y = the number of giant skillets produced
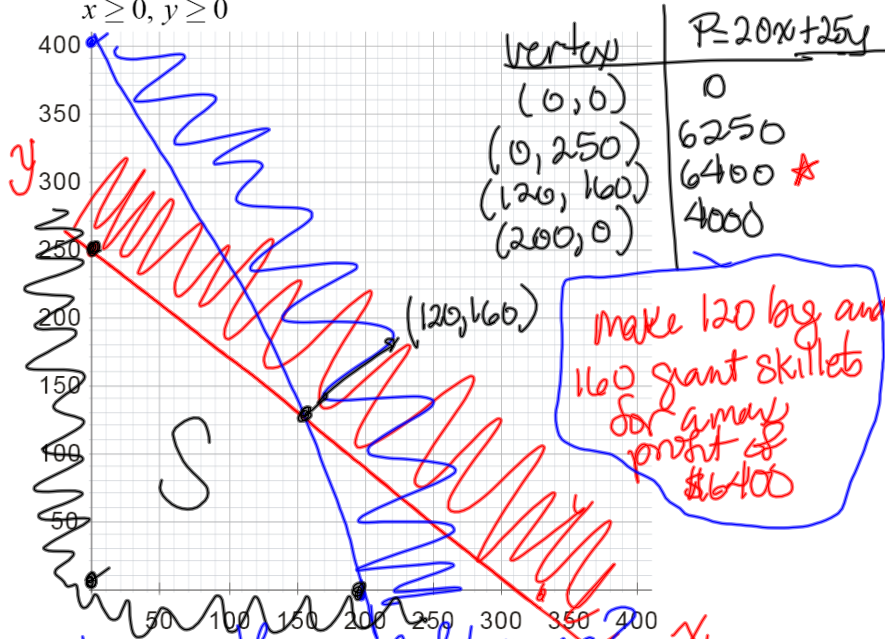P = the profits (in \$) from selling skillets
OBJECTIVE: Maximize $P = 20x + 25y$
SUBJECT TO:

$3x + 4y \leq 1000$ Pounds of iron

$6x + 3y \leq 1200$ Minutes of labor

$x \geq 0, y \geq 0$

Handwritten annotations:

$(0, 250)$ & $(333\frac{1}{3}, 0)$
$(0, 400)$ & $(200, 0)$

vertex	$P = 20x + 25y$
$(0,0)$	0
$(0,250)$	6250
$(120, 160)$	6400 ★
$(200, 0)$	4000

$(120, 160)$

make 120 big and 160 giant skillets for a max profit of \$6400

Is anything left over?

Iron $= 3x + 4y = 3(120) + 4(160) = 1000$
no iron left over

labor $= 6x + 3y = 6(120) + 3(160) = 1200$
no time left over

$P = 12x + 40y$ is our profit

vertex	$P = 12x + 40y$
$(0,0)$	0
$(0,250)$	10000 ★
$(120, 160)$	7840
$(200, 0)$	2400

make 0 big and 250 giant skillets for a profit of \$10,000

Iron $= 3(0) + 4(250)$ all iron used
labor $= 6(0) + 3(250) \leq 1200$

450 min of labor left over

Example

A dietitian is to prepare two foods in order to meet certain requirements. (…) How many ounces of each type of food should be used in order to minimize the cost?
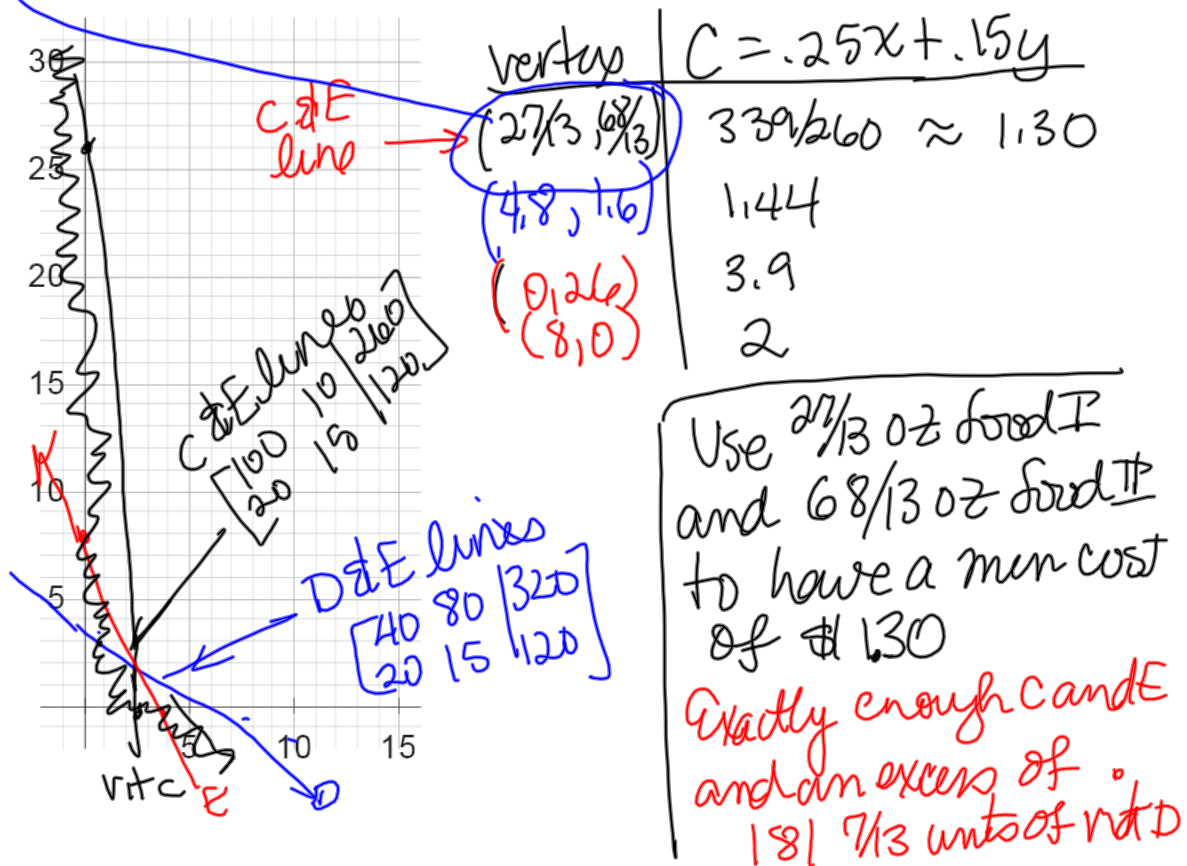
x = the number of ounces of food I
y = the number of ounces of food I
C = the cost (in $) for the food
OBJECTIVE: Minimize $C = 0.25x + 0.15y$
SUBJECT TO:

$100x + 10y \geq 260$ Units of Vit. C $(0,26)$ & $(2.6,0)$

$40x + 80y \geq 320$ Units of Vit. D $(0,4)$ & $(8,0)$

$20x + 15y \geq 120$ Units of Vit. E $(0,8)$ & $(6,0)$

$x \geq 0, y \geq 0$

vertex	$C = .25x + .15y$
$(27/3, 68/3)$	$339/260 \approx 1.30$
$(4.8, 1.6)$	1.44
$(0,26)$	3.9
$(8,0)$	2

C & E line →

C & E lines
$\begin{bmatrix} 100 & 10 & 260 \\ 20 & 15 & 120 \end{bmatrix}$

D & E lines
$\begin{bmatrix} 40 & 80 & 320 \\ 20 & 15 & 120 \end{bmatrix}$

vit C E

Use $27/3$ oz food I and $68/13$ oz food II to have a min cost of $1.30

Exactly enough C and E and an excess of $181 \, 7/13$ units of vit D

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Example

A craftsman has 150 units of wood, 90 units of glue and 150 units of paint.
(…) how many of each should be made to maximize the revenue?

x = the number of small picture frames produced
y = the number of large picture frames produced
C = the from selling picture frames (in dollars)

OBJECTIVE: Maximize $R = 175x + 400y$

SUBJECT TO:
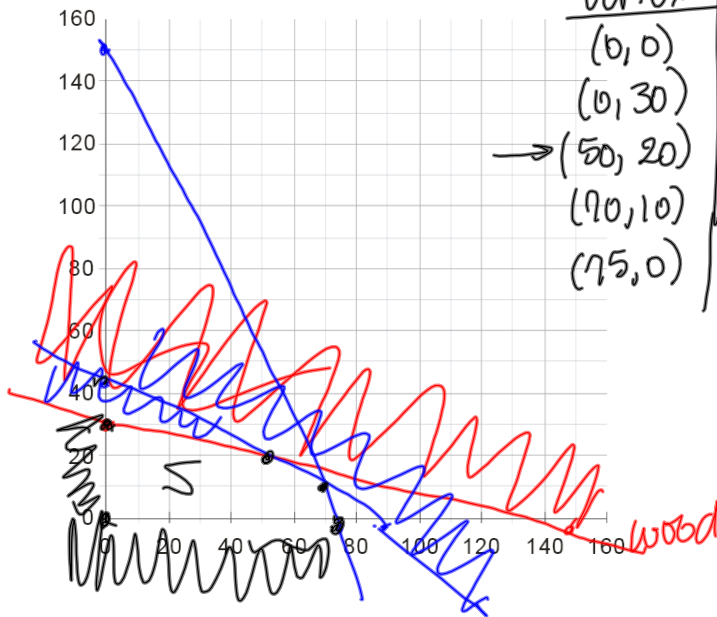
$1x + 5y \leq 150$	Units of wood	$(0, 30)$ & $(150, 0)$
$1x + 2y \leq 90$	Units of glue	$(0, 45), (90, 0)$
$2x + 1y \leq 150$	Units of paint	$(0, 150))$
$x \geq 0, y \geq 0$		

Vertex	$R = 175x + 400y$
$(0, 0)$	0
$(0, 30)$	$12,000$
$\rightarrow (50, 20)$	$16,750$
$(90, 10)$	$16,250$
$(75, 0)$	$13,125$

wood

Make 50 small frames and 20 large frames for a max profit of $16,750

Leftovers?

Post-Optimal
$1(50) + 5(20) = 150$ units of wood $\Rightarrow$ 0 wood leftover
$1(50) + 2(20) = 90$ units of glue used
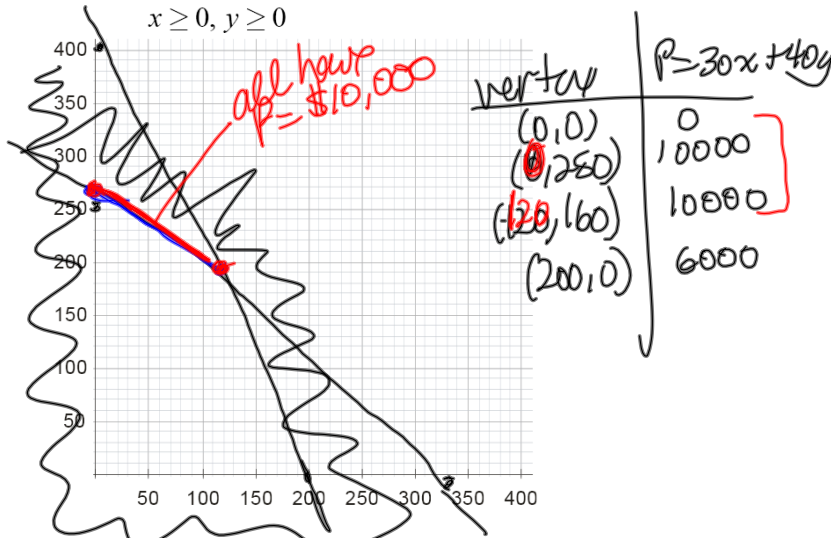$2(50) + 1(20) = 120$ units of paint

All the wood and glue were used.

There were $150 - 120 = 30$ units of paint left

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Example

A company produces handmade skillets in two sizes, big and giant. (…)
how many each size skillet should be made to maximize profits if big
skillets have a profit of $30 each and giant skillets have a profit of $40
each?

x = the number of big skillets produced

y = the number of giant skillets produced

P = the profits (in $) from selling skillets

OBJECTIVE: Maximize $P = 30x + 40y$

SUBJECT TO:

$3x + 4y \le 1000$ Pounds of iron

$6x + 3y \le 1200$ Minutes of labor

$x \ge 0,\ y \ge 0$

abl hour = $10,000

vertex	P=30x+40y
(0,0)	0
(0,250)	10000
(120,160)	10000
(200,0)	6000

$P = 10000 = 30x + 40y$

$y = (-3/4)x + 250$ with $(0 \le x \le 120)$

x = # of big skillets, y = # giant skillets

$= 0, 4, 8, \dots 120$

Leftovers ?

① $x = 0,\ y = 250$
$3(0) + 4(250) = 1000 \Rightarrow 0$ iron left
$6(0) + 3(250) = 750 \Rightarrow 450$ min of labor left

② $x = 40,\ y = -3/4x + 250 = 220$
$3(40) + 4(220) = 1000 \Rightarrow 0$ iron left
$6(40) + 3(220) \Rightarrow 800$ min of labor left

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Example

A dietitian is to prepare two foods in order to meet certain requirements. (…) How many ounces of each type of food should be used in order to minimize the cost if an ounce of food I costs 20 cents and an ounce of food II costs 15 cents?

x = the number of ounces of food I
y = the number of ounces of food I
C = the cost (in $) for the food
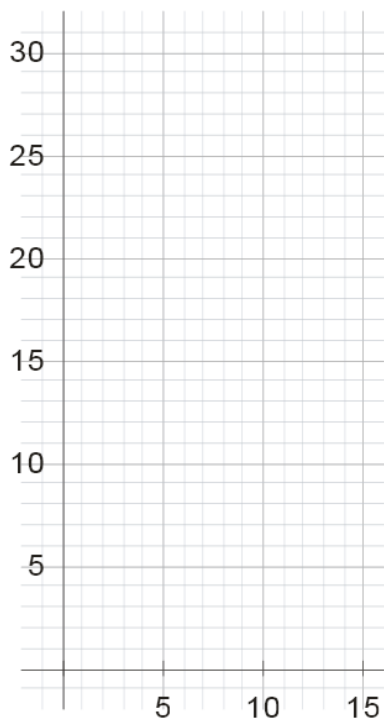OBJECTIVE: Minimize $C = 0.20x + 0.15y$
SUBJECT TO:

$$100x + 10y \geq 260 \qquad \text{Units of Vit. C}$$
$$40x + 80y \geq 320 \qquad \text{Units of Vit. D}$$
$$20x + 15y \geq 120 \qquad \text{Units of Vit. E}$$
$$x \geq 0, \, y \geq 0$$

Vertex	$C = .2x + .15y$
$(0, 26)$	3.90
$(8, 0)$	1.6
$(4.8, 1.6)$	1.20
$(\tfrac{24}{13}, \tfrac{68}{13})$	1.20

$$C = .2x + .15y = 1.20$$
$$y = -\tfrac{4}{3}x + 8$$
$$\tfrac{24}{13} \leq x \leq 4.8$$

on the Vit E line

The min cost of $1.20 occurs when $\tfrac{24}{13} \leq x \leq 4.8$ oz of food I and $y = -\tfrac{4}{3}x + 8$ oz of food II