1 Banach spaces and Hilbert spaces

We will first discuss Banach spaces, since much of what we say applies to Hilbert spaces, without change.

Let V be a normed linear space over either the real or complex numbers. A sequence of vectors $\{v_j\}_{j=1}^{\infty}$ is a map from the natural numbers to V. We say that v_j *converges* to $v \in V$ if

$$\lim_{j\to\infty} \|v_j - v\| = 0.$$

A sequence $\{v_j\}$ is said to be *Cauchy* if for each $\epsilon > 0$, there exists a natural number N such that $\|v_j - v_k\| < \epsilon$ for every $j, k \geq N$. Every convergent sequence is Cauchy, but there are many examples of normed linear spaces V for which there exists non-convergent Cauchy sequences. One such example is the set of rational numbers $\mathbb{Q}$. The sequence $(1.4, 1.41, 1.414, \dots)$ converges to $\sqrt{2}$ which is not a rational number. We say a normed linear space is *complete* if every Cauchy sequence is convergent in the space. The real numbers are an example of a complete normed linear space.

We say that a normed linear space is a Banach space if it is complete. We call a complete inner product space a Hilbert space. Consider the following examples:

1. Every finite dimensional normed linear space is a Banach space. Likewise, every finite dimensional inner product space is a Hilbert space.

2. Let $x = (x_1, x_2, ..., x_n, ...)$ be a sequence. The following spaces of sequences are Banach spaces:

$$\ell^p = \{x : \sum_{j=1}^{\infty} |x_j|^p = \|x\|_{\ell^p} < \infty\}, 1 \leq p < \infty \tag{1.1}$$

$$\ell^\infty = \{x : \sup_j |x_j| < \infty\} \tag{1.2}$$

$$c = \{x : \lim_{j\to\infty} x_j \text{ exists}\}, \|x\|_c = \|x\|_\infty \tag{1.3}$$

$$c_0 = \{x : \lim_{j\to\infty} x_j = 0\}, \|x\|_{c_0} = \|x\|_\infty \tag{1.4}$$

Except for ℓ^∞. the spaces above are separable – i.e., each has a countably dense subset,

3. The following function spaces are Banach spaces:

$$C[0,1],\ \|f\|_{C[0,1]} = \max_{x\in[0,1]} |f(x)| \tag{1.5}$$

$$C^{(k)}[a,b],\ \|f\|_{C^{(k)}[a,b]} = \sum_{j=0}^{k} \sup_{x\in[0,1]} |f^j(x)| \tag{1.6}$$

$$L^p(I),\ \|f\| = \left(\int_I |f(x)|^p\right)^{\frac{1}{p}} \tag{1.7}$$

$$L^\infty(I),\ \|f\|_{L^\infty(I)} = \text{ess-sup}_{x\in I}|f(x)| \tag{1.8}$$

There are two Hilbert spaces among the spaces listed: the sequence space ℓ^2 and the function space L^2. In the result below, we will show that ℓ^∞ is complete. After that, we will show that $C[0,1]$ is complete, relative to the *sup-norm*, $\|f\|_{C[0,1]} = \max|f(x)|$. Of course, this means that both of them are Banach spaces.

Proposition 1.1. *The space ℓ^∞ is a Banach space.*

Proof. The norm on ℓ^∞ is given by

$$\|x\|_\infty = \sup_j |x(j)|.$$

Let $\{x_n\}_{n=1}^\infty \subset \ell^\infty$ denote a Cauchy sequence of elements in ℓ^∞. We show that this sequence converges to $x \in \ell^\infty$. Since $\{x_n\}$ is Cauchy, for each $\epsilon > 0$, there exists an N such that for all $n, m \geq N$,

$$\|x_n - x_m\|_\infty < \epsilon. \tag{1.9}$$

This implies that $|x_n(j) - x_m(j)| < \epsilon$ for all j. Therefore, the sequence $\{x_n(j)\}$ is a Cauchy sequence of real numbers, and hence converges to some value $x(j)$. That is, $\lim_{n\to\infty} x_n(j) = x(j)$ exists. From (1.9), if we choose $\epsilon = 1$, then for all $n, m \geq N$, we have

$$\|x_n - x_m\|_\infty < 1.$$

In particular, it follows that

$$\|x_n\| < 1 + \|x_m\|, \qquad n, m \geq N.$$

Fix m. Then, for all $n \geq N$, $|x_n(j)| \leq \|x_n\|_\infty < 1 + \|x_m\|_\infty$. Letting $n \to \infty$, we see that

$$|x(j)| \leq 1 + \|x_m\|_\infty$$

holds uniformly in j. Therefore, $x \in \ell^\infty$. To complete the proof, we need to show that x_n converges to x in norm. We have

$$|x_n(j) - x_m(j)| < \epsilon \qquad \forall j \in \mathbb{N} \text{ and } n, m \geq N.$$

Let $n \to \infty$. Then, $x_n(j) \to x(j)$ so we have that

$$|x(j) - x_m(j)| < \epsilon \qquad \forall j \in N, m \geq N.$$

Since this holds for all j, it follows that $\|x - x_m\|_\infty < \epsilon$ for all $m \geq N$. Therefore, the sequence x_m converges to $x \in \ell^\infty$. □

Proposition 1.2. *Relative to the sup norm, $C[0,1]$ is complete and is thus a Banach space.*

Proof. Let $\{f_n(x)\}_{n=1}^\infty$ be a Cauchy sequence in $C[0,1]$. Then, for every $\epsilon > 0$, there exists an N such that $\|f_n - f_m\| < \epsilon$ for all $n, m \geq N$. For any fixed $t \in [0,1]$, this implies that

$$|f_n(t) - f_m(t)| < \epsilon \qquad \forall\, m, n \geq N. \tag{1.10}$$

Thus, for t fixed, $\{f_n(t)\}_{n=1}^\infty$ is a Cauchy sequence of real numbers, and so it converges. Define $f(t)$ by the pointwise limit of this sequence:

$$f(t) = \lim_{n \to \infty} f_n(t),\ t \in [0,1]. \tag{1.11}$$

By taking the limit as $m \to \infty$ in (1.10), we see that

$$|f_n(t) - f(t)| \leq \epsilon \qquad \forall\, n \geq N,$$

which holds uniformly for all $t \in [0,1]$. Consequently,

$$\|f_n - f\| = \sup_{t \in [0,1]} |f_n(t) - f(t)| \leq \epsilon, \qquad \forall\, n \geq N,$$

and so $\lim_{n\to\infty} \|f_n - f\| = 0$. What remains is to show that $f \in C[0,1]$. To do that, fix t and let $\epsilon > 0$. Because $f_n \in C[0,1]$, there exists a $\delta > 0$ such that

$$|f_n(t+h) - f_n(t)| < \frac{\epsilon}{3} \qquad \forall\, |h| < \delta,$$

assuming, of course, that $t+h \in [0,1]$. Then applying the triangle inequality gives

$$|f(t+h) - f(t)| \le |f(t+h) - f_n(t+h)| + |f_n(t+h) - f_n(t)| + |f_n(t) - f(t)|.$$

From (1.11), we may choose n so large that both $|f(t+h) - f_n(t+h)| < \frac{\epsilon}{3}$ and $|f(t) - f_n(t)| < \frac{\epsilon}{3}$. Once we have found an n such that this holds, we note that for these n

$$|f(t+h) - f(t)| < \frac{\epsilon}{3} + |f_n(t+h) - f_n(t)| + \frac{\epsilon}{3}.$$

Since the f_n are continuous functions, we can find a δ such that $|f_n(t+h) - f(t)| < \frac{\epsilon}{3}$ whenever $|h| < \delta$. It follows that, for this choice of δ, we have

$$|f(t+h) - f(t)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $|h| < \delta$, which implies that $f \in C[0,1]$. □

Just because a space is complete relative to one norm does not mean that the same space will also be complete in another. The example below illustrates this for the space of continuous functions, $C[0,1]$.

Example 1.1. If we replace the sup norm on $C[0,1]$ with the L^1 norm, then $C[0,1]$ is not complete.

Proof. To simplify the discussion, we will work with $[-1,1]$ rather than $[0,1]$. Consider the sequence of continuous functions $f_n \in C[-1,1]$ defined piecewise by

$$f_n(t) = \begin{cases} -1 & x \in [-1, -\frac{1}{n}] \\ nx & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & x \in [\frac{1}{n}, 1]. \end{cases}$$

Let $n > m$. We note that $|f_n(t) - f_m(t)|$ will be symmetric about $t = 0$. Using this, we see that

$$|f_n(t) - f_m(t)| = \begin{cases} nt - mt & t \in [0, \frac{1}{n}] \\ 1 - mt & t \in [\frac{1}{n}, -\frac{1}{m}] \\ 0 & t \in [\frac{1}{m}, 1] \end{cases}$$

Computing the integrals over $[0, 1]$ yields

$$\begin{aligned}\int_0^1 |f_n(t) - f_m(t)|\, dt &= \int_0^{\frac{1}{n}} (n-m)t\, dx + \int_{\frac{1}{n}}^{\frac{1}{m}} 1 - mt\, dx \\ &= (n-m)\frac{1}{2n^2} + \frac{1}{m} - \frac{1}{n} - \frac{m}{2}\frac{1}{m^2} + \frac{m}{2}\frac{1}{n^2} \\ &= \frac{1}{2}\Big[\frac{1}{m} - \frac{1}{n}\Big]\end{aligned}$$

Making use of the symmetry in conjunction with the result above then gives us $\|f_n - f_m\|_{L^1[-1,1]} = \frac{1}{m} - \frac{1}{n}$. Let $N > \frac{2}{\epsilon}$. Then, for $m, n \geq N$, we find that

$$\|f_n - f_m\|_{L^1[-1,1]} < \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so the sequence is Cauchy in the $L^1[-1, 1]$ norm. However, the limit is not continuous. A computation shows that, for the step function $f(t)$ defined by

$$f(t) = \begin{cases} -1 & t \in [-1, 0) \\ 0 & t = 0 \\ 1 & t \in (0, 1], \end{cases}$$

we have $\|f - f_n\|_{L^1[-,1]} = \frac{1}{n}$, and so

$$\lim_{n\to\infty} \|f - f_n\|_{L^1} = 0.$$

Therefore, the sequence of functions $f_n \in C[-1, 1]$ converges to a discontinuous function under the $L^1[-1, 1]$ norm, and, consequently, $C[-1, 1]$ is not complete under the $L^1[-1, 1]$ norm. □

2 Approximating Continuous Functions

Modulus of continuity Every $f \in C[0, 1]$ is uniformly continuous: for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(t_2) - f(t_1)| < \epsilon \tag{2.1}$$

as long as $t_1, t_2 \in [0, 1]$ satisfy $|t_2 - t_1| < \delta$.

Using this fact, we are able to make the following definition:

Definition 2.1. Let $f \in C[0,1]$. The *modulus of continuity* for f is defined to be

$$\omega(f,\delta) = \sup_{|t_1 - t_2| \leq \delta} |f(t_1) - f(t_2)|, \ \forall\, t_1, t_2 \in [0,1]. \tag{2.2}$$

Example 2.1. Let $f(x) = \sqrt{x}$, $0 \leq x \leq 1$. Show that $\omega(f,\delta) \leq C\sqrt{\delta}$.

Proof. Let $0 < s < t \leq 1$. We note that

$$\begin{aligned} 0 < \sqrt{t} - \sqrt{s} &= \frac{t-s}{\sqrt{t}+\sqrt{s}} \\ &= \sqrt{t-s}\left(\frac{\sqrt{t-s}}{\sqrt{t}+\sqrt{s}}\right) \\ &= \sqrt{t-s}\frac{\sqrt{1-\frac{s}{t}}}{1+\sqrt{\frac{s}{t}}} \\ &\leq \sqrt{t-s} = \sqrt{\delta} \end{aligned}$$

Hence, $\omega(f,\delta) \leq \sqrt{\delta}$. To get equality, take $s = 0$ and $t = \delta$. □

Example 2.2. Suppose that $f \in C^{(1)}[0,1]$. Show that $\omega(f,\delta) \leq ||f'||_\infty \delta$.

Proof. Because $f \in C^{(1)}$, we can estimate $f(t) - f(s)$ this way:

$$|f(t) - f(s)| \leq \int_s^t |f'(x)|\, dx \leq (t-s)\|f'\|_\infty \leq \delta \|f'\|_\infty, \tag{2.3}$$

which immediately gives $\omega(f,\delta) \leq \delta\|f'\|_\infty$. □

Linear splines One very effective way to approximate a continuous function $f \in C[0,1]$, given values of f at a finite set of points in $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$, is to use a "connect-the-dots" interpolant. The interpolant is obtained by joining points $(t_j, f(t_j))$ by a straight. This procedure results in a piecwise-linaer, continuous function with corners at the the t_j's. More formally, this is a called *linear spline* interpolant. Linear spline interpolants are used for generating plots in many standard programs, such as Matlab or Mathematica.

Defining a space of linear splines starts with sequence of points (or partition of $[0,1]$) $\Delta = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}$. Δ called a *knot* *sequence*. Linear splines on $[0,1]$ with knot sequence Δ are the set of all piecewise linear functions that are continuous on $[0,1]$ and (possibly) have corners at the knots. As described above, we can interpolate continuous functions using linear splines. Let $f \in C[0,1]$ and let $y_j = f(t_j)$. This is linear spline $s_f(x)$ that is constructed by joining pairs of points (t_j, y_j) and (t_{j+1}, y_{j+1}) with straight lines; the resulting spline is the unique. The result below gives an estimate of the error made by replacing f by s_f.

Proposition 2.1. *Let $f \in C[0,1]$ and let $\Delta = \{t_0 = 0 < t_1 < \cdots < t_n = 1\}$ be a knot sequence with norm $\|\Delta\| = \max |t_j - t_{j+1}|$, $j = 0, \ldots, n-1$. If s_f is the linear spline that interpolates f at the t_j's, then,*

$$\|f - s_f\|_\infty \le \omega(f, \|\Delta\|) \tag{2.4}$$

Proof. Consider the interval $I_j = [t_j, t_{j+1}]$. We have on I_j that $s_f(t)$ is a line joining $(t_j, f(t_j))$ and $(t_{j+1}, f(t_{j+1}))$; it has the form

$$s_f(t) = \frac{t_{j+1} - t}{t_{j+1} - t_j} f(t_j) + \frac{t - t_j}{t_{j+1} - t_j} f(t_{j+1})$$

Also, note that we have

$$\frac{t_{j+1} - t}{t_{j+1} - t_j} + \frac{t - t_j}{t_{j+1} - t_j} = 1.$$

Using these equations, we see that $f(t) - s_f(t)$ for any $t \in [t_j, t_{j+1}]$ can be written as

$$\begin{aligned} f(t) - s_f(t) &= f(t)\big(\frac{t_{j+1} - t}{t_{j+1} - t_j} + \frac{t - t_j}{t_{j+1} - t_j}\big) - s_f(t) \\ &= (f(t) - f(t_j))\frac{t_{j+1} - t}{t_{j+1} - t_j} + (f(t) - f(t_{j+1}))\frac{t - t_j}{t_{j+1} - t_j}. \end{aligned}$$

By the definition of the modulus of continuity, $|f(x) - f(y)| \le \omega(f, \delta)$ for any x, y such that $|x - y| \le \delta$. If we set $\delta_j = t_{j+1} - t_j$, then we see that on the interval I_j we have

$$\begin{aligned} |f(t) - s_f(t)| &\le \big|(f(t) - f(t_j))\big|\frac{t_{j+1} - t}{t_{j+1} - t_j} + \big|f(t) - f(t_{j+1})\big|\frac{t - t_j}{t_{j+1} - t_j} \\ &\le \left(\frac{t_{j+1} - t}{t_{j+1} - t_j} + \frac{t - t_j}{t_{j+1} - t_j}\right)\omega(f, \delta_j) = \omega(f, \delta_j). \end{aligned}$$

Because the modulus of continuity is non decreasing (exercise 4.5(c)) and $\delta_j \leq \|\Delta\|$, we have $\omega(f, \delta_j) \leq \omega(f, \|\Delta\|)$. Consequently, $|f(t) - s_f(t)| \leq \omega(f, \|\Delta\|)$, uniformly in t. Taking the supremum on the right side of this inequality then yields (2.4). □

3 Basis Splines – B-Splines

For notation, we define

$$(x)_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and

$$N_2(x) = (x)_+ - 2(x-1)_+ + (x-2)_+. \tag{3.1}$$

Let Δ be an equally spaced knot sequence $t_j = \frac{j}{n}$, $j = 0, \ldots, n$.

Proposition 3.1. *Let $B = \{N_2(nx - j + 1) : j = 0, \ldots, n\}$. Then, B is a basis for $S^{\frac{1}{n}}(1, 0)$ provided $x \in [0, 1]$.*

Proof. Exercise. □

Example 3.1. Consider $n = 4$. Recall that the values at the corners and endpoints determine the linear spline. So, let y_j be given at $j = 0, 1, 2, 3, 4$. Then, the interpolating spline is

$$s(x) = \sum_{j=0}^{4} y_j N_2(4x - j + 1), \qquad 0 \leq x \leq 1.$$

Theorem 3.1. *There is a countably dense set of continuous functions.*

Proof. Consider the functions of the form

$$\sum_{j=0}^{n} q_j N_2(nx - j + 1), \qquad x \in [0, 1], q_j \in \mathbb{Q}. \tag{3.2}$$

For each n, there are countably many such splines because the rationals are a countable set. Since there are countably many n and the union of countably many countable sets is countable, the collection of such functions is countable. We now demonstrate that this is a dense subset of $C[0,1]$. Let $f \in C[0,1]$ and consider the interpolant

$$s_f(x) = \sum_{j=0}^{n} f(\frac{j}{n}) N_2(nx - j + 1$$

for $x \in [0,1]$. Then, we have previously shown that $\|f - s_f\|_\infty \leq \omega(f, \frac{1}{n})$. Consider another spline

$$\tilde{s}(x) = \sum_{j=0}^{N} q_j N_2(nx - j + 1) \tag{3.3}$$

which is a member of the countable set of splines we defined previously. By computation, it follows that

$$s(x) - \tilde{s}(x) = \left(\frac{\frac{j+1}{n} - x}{\frac{1}{n}}\right)\left(f\big(\frac{j}{n}\big) - q_j\right) + \left(\frac{x - \frac{j}{n}}{\frac{1}{n}}\right)\left(f\big(\frac{j+1}{n}\big) - q_{j+1}\right).$$

Therefore, we can bound the maximum pointwise error by

$$\|s - \tilde{s}\|_\infty \leq \max_{0 \leq j \leq n} |f(\frac{j}{n}) - q_j|.$$

Choose each q_j to be a rational approximation of $f(\frac{j}{n})$. That is, we choose $q_j \in \mathbb{Q}$ such that $|q_j - f(\frac{j}{n})| < \frac{\epsilon}{2}$ for each $j = 0, \ldots, n$. Then, it follows that $\|s - \tilde{s}\|_\infty < \frac{\epsilon}{2}$. By choosing n large enough so that $\omega(f, \frac{1}{n}) < \frac{\epsilon}{2}$ and the q_j in this manner, we find by applying the triangle inequality that

$$\begin{aligned} \|f - \tilde{s}\|_\infty &\leq \|f - s\|_\infty + \|s - \tilde{s}\|_\infty \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, we have constructed a countable set of splines and demonstrated that they are dense in $C[0,1]$. □

Note that both ℓ^∞ and L^∞ are not separable. In ℓ^∞, consider the uncountable set $A = \{x \in \ell^\infty : x_j \in \{0,1\}\}$. If $x_\alpha, x_{\alpha'} \in A$ differ, then $\|x_\alpha - x_{\alpha'}\|_\infty = 1$, since they differ in at least one coordinate j, and the only possibility is for $|x_\alpha(j) - x_{\alpha'}(j)| = 1$. Suppose that ℓ^∞ is separable. Then, there exists a countable dense subset $\{v_j\}_{j=1}^{\infty}$. In particular, for any $x \in \ell^\infty$ and for each ϵ, there exists at least one v_j such that $\|v_j - x\| < \epsilon$. Let $\epsilon = \frac{1}{4}$. Since there are uncountably many $x_\alpha \in A$, there exists a v_j such that there exists $x_\alpha \neq x_{\alpha'} \in A$ such that both $|v_j - x_\alpha| < \epsilon$ and $|v_j - x_{\alpha'}| < \epsilon$. Then, by the triangle inequality, we find

$$\begin{aligned} \|x_\alpha - x_{\alpha'}\|_\infty &\leq \|x_\alpha - v_j\|_\infty + \|x_{\alpha'} - v_j\|_\infty \\ &< \epsilon + \epsilon = \frac{1}{2} \end{aligned}$$

which implies that $\|x_\alpha - x_{\alpha'}\|_\infty < \frac{1}{2}$, which is a contradicts the computation $\|x_\alpha - x_{\alpha'}\|_\infty = 1$.

4 Finite Element Spaces

Let $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$. We refer to this as a *knot sequence*, which we denote by $\Delta = \{t_j\}_{j=0}^{n}$. We define the subintervals $I_j = [t_j, t_{j+1})$ and let π_k denote the set of polynomials of degree less than or equal to k. We define the space of splines as follows:

$$S^\Delta(k, r) = \{\phi : [0,1] \to \mathbb{R} : \phi|_{I_j} \in \pi_k(I_j) \text{ and } \phi \in C^{(r)}([0,1])\} \tag{4.1}$$

When $r = -1$, ϕ is discontinuous. The finite element spaces $S^{\frac{1}{n}}(k, r)$ are degree k polynomials on each interval and have $r \leq k-1$ derivatives that match at the interior points. We consider the following question: how many parameters are required to describe a function in $S^{\frac{1}{n}}(k, r)$? That is, what is the dimension of this linear space?

There are n intervals and on each interval and on each interval there are $k+1$ free parameters since the function is a degree k polynomial on each interval. Therefore, we have $n(k+1)$ free parameters. At each of the $n-1$ knots, the polynomials most smoothly join, there are $r+1$ equations that must match (the polynomials across a knot must match and their r derivatives must match). This yields $(n-1)(r+1)$ constraints. Therefore, we have at least $n(k+1) - (n-1)(r+1) = n(k-r) + r + 1$ parameters. It follows that the dimension of $S^{\frac{1}{n}}(k, r) = n(k-r) + r + 1$ provided that the equations at the knots are independent (which can be shown).

For an example, consider $k = 1, r = 0$. This is the space $S^{\frac{1}{n}}(1, 0)$ which has dimension $n(1-0)+0+1 = n+1$. If we consider $k = m-1$, $r = m-2$, then the dimension $S^{\frac{1}{n}}(m-1, m-2)$ is $n(m-1-m+2)+m-2+1 = n+m-1$.