Midterm Test

Instructions. This test is due on 10/29/13. You may get help on the test *only* from your instructor, and no one else. You *may* use other books, the web, etc. If you do so, quote the source.

1. **(10 pts.)** Use the Courant-Fischer mini-max theorem to show that $\lambda_2 < 0$ for the matrix
$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

2. **(15 pts.)** Let A be an $n \times n$ complex matrix that satisfies $A^*A = AA^*$. Show that A is diagonalizable and that there is a unitary matrix U for which $U^*AU = \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

3. Let f be continuous on $[0,1]$, with $f(0) = f(1) = 0$ and let $s \in S^{1/n}(1,0)$ be the linear spline interpolant to s, with knots at $x_j = \frac{j}{n}$.

   (a) **(15 pts.)** Let $\lambda \in \mathbb{R}$. Show that $\left| \int_0^1 s(x)e^{i\lambda x}dx \right| \le \frac{2n^2}{\lambda^2}\omega(f, 1/n)$.

   (b) **(5 pts.)** Use the previous part to show that $\left| \int_0^1 f(x)e^{i\lambda x}dx \right| \le \omega(f, 1/n) + \frac{2n^2}{\lambda^2}\omega(f, 1/n)$.

4. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal with respect to a weight function $w(x)$ on a domain $[a,b]$. Assume that the degree of ϕ_n is n, and that coefficient of x^n in $\phi_n(x)$ is $k_n > 0$. In addition, suppose that the continuous functions are dense in $L^2_w[a,b] = \{f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 w(x)dx < \infty\}$.

   (a) **(5 pts.)** Show that ϕ_n is orthogonal to all polynomials of degree $n-1$ or less.

   (b) **(10 pts.)** Show that $\{\phi_n\}_{n=0}^{\infty}$ is complete in $L^2_w[a,b]$.

   (c) **(10 pts.)** Show that the polynomials satisfy the recurrence relation $\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n\phi_{n-1}(x)$. Find A_n in terms of the k_n's.

5. Suppose that $f(\theta)$ is 2π-periodic function in $C^m(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k^{th} (complex) Fourier coefficient for f, and let $c_k^{(j)}$ denote the k^{th} (complex) Fourier coefficient for $f^{(j)}$.

   (a) **(5 pts.)** Show that $c_k^{(j)} = (ik)^j c_k$, $j = 1, \ldots, m+1$.

   (b) **(5 pts.)** For $k \neq 0$, show that the Fourier coefficient c_k satisfies the bound
   $$|c_k| \leq \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L^1[0,2\pi]}.$$

   (c) **(10 pts.)** Let $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \geq 1$. Show that both of these hold for f.
   $$\|f - S_n\|_{L^2} \leq C \frac{\|f^{(m+1)}\|_{L^1}}{n^{m+\frac{1}{2}}} \text{ and } \|f - S_n\|_{C[0,2\pi]} \leq C' \frac{\|f^{(m+1)}\|_{L^1}}{n^m}$$

   (d) **(10 pts.)** Let $f(x)$ be the 2π-periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Verify that f satisfies the conditions above with $m = 1$. With the help of (a), calculate the Fourier coefficients for f. (Hint: look at f''.)