Final Exam

Instructions. This test is due on 12/11/13. You may get help on the test *only* from your instructor, and no one else. You *may* use my notes, other books, the web, etc. If you do so, quote the source. Use this test paper as a cover sheet when you turn in the exam.

Notation. $\mathcal{H}$ denotes a separable Hilbert space; $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$; $\mathcal{C}(\mathcal{H})$, the compact operators in $\mathcal{B}(\mathcal{H})$.

1. Let $f \in C[0, 2\pi]$. In assignment 5, if you scale the integrals involved, you derived the trapezoidal rule for numerically approximating $\int_0^{2\pi} f(x)dx$. If $a = 0$, $b = 2\pi$ and f is 2π periodic, this has the form $Q_{trap}(f) = \frac{2\pi}{n}\big(\sum_{k=0}^{n-1} f(\frac{2\pi k}{n})\big)$. Let $E_n = \big|\int_0^{2\pi} f(x)dx - Q_n(f)\big|$.

   (a) **(5 pts.)** Show that $Q_{trap}(e^{ikx}) = \begin{cases} 0 & k \not\equiv 0 \bmod n \\ 2\pi & k \equiv 0 \bmod n \end{cases}$.

   (b) **(10 pts.)** Let $f(x)$ be the 2π-periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Show that $\int_0^{2\pi} f(x)dx = 16\pi^5/15$. Prove that $E_n \le Cn^{-4}$. (Hint: $f = \frac{8\pi^4}{15} - \frac{24}{\pi}\sum_{k\neq 0} e^{ikx}k^{-4}$.)

   (c) **(5 pts.)** Use Matlab or some other program to plot $\log(E_n)$ vs. $\log n$, for $n = 16, 64, 256, 1024$. This should be a straight line. What is its slope? Does it agree with what you found in part (b)?

2. Let $\mathcal{H}$ be a *complex* Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$., and let $L \in \mathcal{B}(\mathcal{H})$

   (a) **(5 pts.)** Verify that $\langle L(u + e^{i\alpha}v), u + e^{i\alpha}v\rangle - \langle L(u - e^{i\alpha}v), u - e^{i\alpha}v\rangle = 2e^{-i\alpha}\langle Lu, v\rangle + 2e^{i\alpha}\langle Lv, u\rangle$. (You will need this below.)

   (b) **(10 pts.)** Show that if $L = L^*$, then $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$.

   (c) **(10 pts.)** Show that if $M = \sup_{\|u\|=1} |\langle Lu, u\rangle$, then $M \le \|L\| \le 2M$, whether or not L is self adjoint. Give an example that shows this result is *false* in a *real* Hilbert space.

3. **(10 pts.)** Let $K \in \mathcal{C}(\mathcal{H})$ be self adjoint. Show that the only possible limit point of the set of eigenvalues of K is 0; i.e., the non-zero eigenvalues of K are isolated.

4. **(5 pts.)** Let $K \in \mathcal{C}(\mathcal{H})$ be self adjoint. Suppose the range of K contains a dense subset of $\mathcal{H}$, and that an o.n. basis has been chosen for the eigenspace of each nonzero eigenvalue of K. Show that the set of all of these eigenvectors form a complete, orthonormal set.

5. Let $\mathcal{H} = L^2[0,1]$. Consider the boundary value problem,

$$Lu := \frac{d}{dx}\left((1+x)\frac{du}{dx}\right) = f(x),\ u(0) = 0,\ u'(1) = 0. \tag{1}$$

   (a) **(5 pts.)** Find $G(x,y)$, the Green's function for (1).

   (b) **(5 pts.)** Let $Gf(x) = \int_0^1 G(x,y)f(y)dy$. Show that the range of G contains a dense set in $\mathcal{H}$. (Hint: show that every $v \in C^2[0,1]$ with support in $(0,1)$ is in the range of G. Explain why this is dense in $\mathcal{H}$.)

   (c) **(5 pts.)** Use it and the previous problem to show that the eigenfunctions for $\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) + \lambda u = 0$, $u(0) = 0$, $u'(1) = 0$ form a complete orthogonal set.

6. Let $\|\cdot\|_{op}$ be the operator norm for $\mathcal{B}(\mathcal{H})$.

   (a) **(5 pts.)** Show that in $\|\cdot\|_{op}$, then $\mathcal{B}(\mathcal{H})$ is a Banach space.

   (b) **(10 pts.)** Consider the operator $L = I - \lambda M$, with $M \in \mathcal{B}(\mathcal{H})$. Show that if $|\lambda| < \|M\|_{op}^{-1}$, then, in the operator norm, $\sum_{k=0}^{\infty} \lambda^k M^k = (I - \lambda M)^{-1}$. (This series is called a Neumann expansion for $(I - \lambda M)^{-1}$.)

7. **(10 pts.)** Show that B, B^{-1} are in $\mathcal{B}(\mathcal{H})$, and $K \in \mathcal{C}(\mathcal{H})$, then the range of $L = B + \lambda K$ is closed.