Test II – Key

Instructions: *Show all work in your bluebook. Calculators that do linear algebra or calculus are* not *allowed.*

1. Define each space listed and describe the operations of vector addition (+) and scalar multiplication ($\cdot$) corresponding to it.

   (a) **(5 pts.)** $\mathcal{P}_n$ is the set of all polynomials of degree n or less; that is, $\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_nx^n\}$. Here are the operations. If $p, q \in \mathcal{P},\ p(x) = a_0 + a_1x + \cdots + a_nx^n\ \ q(x) = b_0 + a_1x + \cdots + b_nx^n$, then
   $(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$.
   If c is a scalar, then $c \cdot p$ is the polynomial
   $(c \cdot p)(x) = ca_0 + ca_1x + \cdots + ca_nx^n$.

   (b) **(5 pts.)** $C^{(1)}[0, 1]$ is the set of all functions f defined and continuously differentiable on the interval $[0, 1]$. If $f, g \in C^{(1)}[0, 1]$, then $f + g$ is defined by
   $(f + g)(x) = f(x) + g(x)$
   and $c \cdot f$ is defined by
   $(c \cdot f)(x) = cf(x)$.

2. **(15 pts.)** Determine whether or not the set S of 2×2 matrices $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $x + w = 0$ is a subspace of $\mathcal{M}_{2,2}$.

   Solution. Is $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in S? Yes, since $x + w = 0 + 0 = 0$. Is S closed under addition? Let $M_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix}$ be in S. We need to check whether $(x_1+x_2)+(w_1+w_2)$ is 0. Rearranging terms, we get $(x_1 + x_2) + (w_1 + w_2) = (x_1 + w_1) + (x_2 + w_2) = 0 + 0 = 0$. Thus, $M_1 + M_2$ is in S. Is S closed under scalar multiplication? To check this, we to see whether $cx + cw = 0$. Again, this is true because $cx + cw = c(x + w) = c \cdot 0 = 0$

3. **(15 pts.)** Determine whether or not the set $\{1, e^x, e^{2x}\}$ is linearly independent in $C(-\infty, \infty)$.

Solution. Start with the equation $c_1 + c_2e^x + c_3e^{2x} \equiv 0$. Differentiate this twice to get $c_2e^x + 2c_3e^{2x} \equiv 0$ and $c_2e^x + 4c_3e^{2x} \equiv 0$. Set $x = 0$ in the three equations. This results in the system

$$c_1 + c_2 + c_3 = 0, \quad c_2 + 2c_3 = 0, \quad c_2 + 4c_3 = 0$$

Subtracting the second equation from the third gives $2c_3 = 0$, so $c_3 = 0$. Using this in the second equation gives $c_2 + 2 \cdot 0 = 0$, so $c_2 = 0$. Using both values in the first equation then gives $c_1 = 0$. It follows that the set is linearly independent.

4. **(10 pts.)** Consider $G : C(-\infty, \infty) \to C(-\infty, \infty)$ given by $Gu(x) = \int_0^x e^t u(t)dt$. Show that G is linear and that it is one-to-one.

Solution. By inspection, the domain and range of G are vector spaces. Also, by rules from algebra and calculus, we have:

$$\begin{aligned} G[u+v](x) &= \int_0^x e^t(u(t) + v(t))dt \\ &= \int_0^x e^t u(t)dt + \int_0^x e^t u(t)dt \\ &= Gu(x) + Gv(x). \end{aligned}$$

Thus G is *additive.* In addition, if c is a scalar, then we have:

$$\begin{aligned} G[cu](x) &= \int_0^x e^t(cu(t))dt \\ &= c\int_0^x e^t u(t)dt \\ &= cGu(x). \end{aligned}$$

Thus, G is also *homogeneous.* G thus satisfies the conditions for it to be linear. To see that G is one-to-one, we need to solve for u when $Gu(x) \equiv 0$. The fundamental theorem of calculus implies

$$\frac{d}{dx}\left(\int_0^x e^t(cu(t))dt\right) = e^x u(x) \equiv 0$$

Dividing by e^x then gives us that $u = 0$. This is equivalent to a linear function being one-to-one, so G is one-to-one.

5. **(20 pts.)** Find bases for the column space, null space, and row space of C, and state the rank and nullity of C. What should these sum to? Do they?

$$C = \begin{pmatrix} 1 & -3 & -1 & -3 \\ -1 & 3 & 2 & 4 \\ 2 & -6 & 4 & 0 \end{pmatrix}$$

Solution. Use row operations to put C in reduced row echelon form.

$$C \Longleftrightarrow R = \begin{pmatrix} 1 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By the various methods described in class, the bases for the column space, row space, and null space are, respectively, follows.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \right\}, \quad \{ \begin{pmatrix} 1 & -3 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \},$$

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

The rank and nullity of C are both 2. There sum should be 4, which is the number of columns, and it is.

6. Given that $L : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by $L(p) = x^2 p'' - 2(x-1)p' + 3p$ is a linear transformation, do the following:

   (a) **(10 pts.)** Find the matrix of L relative to the basis $B = \{1, x, x^2\}$.
   Solution. First we apply L to the basis. $L[1] = 3$, $L[x] = x + 2$, and $L[x^2] = 2x^2 - 4x^2 + 4x + 3x^2 = x^2 + 4x$. The matrix for L then has as columns the coordinate vectors for each of these, and they are in the same order as B; hence, the matrix is

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) **(5 pts.)** Find $[2 - x + x^2]_B$ and use the matrix from part 6a to solve $L(p) = 2 - x + x^2$ for p.

Solution. First, we have

$$[2 - x + x^2]_B = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The differential equation is completely equivalent to the matrix equation $A[p]_B = [2 - x + x^2]_B$. Let's put this in augmented form and row reduce it.

$$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Hence, $[p]_E = (4 \ -5 \ 1)^T$, and so $p(x) = 4 - 5x + x^2$.

7. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$.

(a) **(10 pts.)** Find the eigenvalues and eigenvectors of A.

Solution. The characteristic polynomial is

$$\begin{aligned} p_A(\lambda) &= \det\begin{pmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{pmatrix} \\ &= \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4), \end{aligned}$$

and so there two eigenvalues, $\lambda 3$ and $\lambda = 4$. The two systems we need to solve to get the eigenvectors are just

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right)$$

These give the eigenvectors $(-1 \ 1)^T$ and $(-1 \ 2)^T$, for 3, 4, respectively.

(b) **(5 pts.)** Use the answer to part 7a to solve $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.

Solution $\mathbf{x} = c_1 e^{3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$