Notes for April 8, 2003

Last time: We discussed Mallat's multiresolution analysis (MRA), gave some examples, and covered Theorem 5.9. We concluded with a formula for the wavelet, ψ. Before reading these notes, you should review what we covered.

The wavelet. The formula for the wavelet is constructed from the two-scale relation,

$$\phi(x) = \sum_k p_k \phi(2x - k). \tag{1}$$

Here, to keep things simple, we will assume that the p_k's are all real, and that only p_0, p_1, p_2, p_3 are different from 0. That is, we will assume that

$$\phi(x) = \sum_{k=0}^{3} p_k \phi(2x - k) = p_0\phi(2x) + p_1\phi(2x-1) + p_2\phi(2x-2) + p_3\phi(2x-3)$$

If we translate $\phi(x)$ by ℓ units (right, if $\ell > 0$, or left, if $\ell < 0$), then translated function is $\phi(x - \ell)$. These are important because the set $\{\phi(x-\ell)\}_{\ell\in\mathbb{Z}}$ is an orthonormal[1] basis for the space V_0.

The p_k's are not just random numbers. Theorem 5.9 implies that they satisfy at least four conditions, which we will explicitly write out for our case.

1. $p_{0-2\ell}p_0 + p_{1-2\ell}p_1 + p_{2-2\ell}p_2 + p_{3-2\ell}\, p_3 = \delta_{\ell,0}$, any integer ℓ.
2. $p_1^2 + p_2^2 + p_2^2 + p_3^2 = 2$
3. $p_0 + p_1 + p_2 + p_3 = 2$
4. $p_0 + p_2 = 1$ and $p_1 + p_3 = 1$.

These conditions are not independent. For example, setting $\ell = 0$ in the first implies the second. Moreover, since we are assuming $p_k = 0$ for $k < 0$ and $k > 3$, only $\ell = \pm 1$ results in an equation which isn't identically 0. In fact,

[1] To even use the word "orthonormal" requires an inner product. The one that we are using here is that of $L^2(\mathbb{R})$ for real-valued functions; namely,

$$\langle g, h\rangle := \int_{-\infty}^{\infty} g(x)h(x)dx.$$

these two values of ℓ result in the *same* equation, $p_0p_2+p_1p_3 = 0$. Finally, the fourth condition obviously implies the third. For the present, the conditions that concern us are

$$p_1^2 + p_2^2 + p_2^2 + p_3^2 = 2 \quad \text{and} \quad p_0p_2 + p_1p_3 = 0.$$

Once we know $\phi(x)$, we know an orthonormal basis for all of the spaces V_j. In particular, we know that $\{\sqrt{2}\phi(2x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_1. The wavelet space W_0 is defined to be all functions in V_1 that are orthogonal to the entire space V_0. Symbolically, $W_0 = \{w \in V_1 \colon \langle w, f\rangle = 0 \text{ for all } f \in V_0\}$. Our aim is to construct a function $\psi(x)$ such that the set $\{\psi(x-m)\}_{m\in\mathbb{Z}}$ is an orthonormal basis for the space W_0. To do this, we use our basis for V_1 to expand ψ,

$$\psi(x) = \sum_{k\in\mathbb{Z}} q_k\phi(2x-k),$$

and then we take the inner product of ψ with

$$\phi(x-\ell) = \sum_{k=0}^{3} p_k\phi(2x-k-2\ell) = \sum_{k=2\ell}^{2\ell+3} p_{k-2\ell}\phi(2x-k)$$

and set the result to 0, obtaining

$$\langle\psi(x), \phi(x-\ell)\rangle = 2p_0q_{2\ell} + 2p_1q_{2\ell+1} + 2p_2q_{2\ell+2} + 2p_3q_{2\ell+3} = 0\,. \qquad (2)$$

Let's look at $\ell = 0$. In that case, we have

$$p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 = 0$$

There is a basic trick to finding q_k's that satisfy the equation. Write the p_k's backward and alternate signs. That is,

p's	p_0	p_1	p_2	p_3
q's	p_3	$-p_2$	p_1	$-p_0$

Taking the inner product of the two vectors amounts to multiplying the vertical entries and adding the result. Notice there is a cancellation that occurs among the outer pairs and inner pairs, giving 0 overall. This suggests that we use $q_0 = p_3$, $q_1 = -p_2$, $q_2 = p_1$, $q_3 = -p_0$, and $q_k = 0$ otherwise.

Let's check the $\ell = 1$ case. From (2) and our choice of q's, we have

$$\langle \psi(x), \phi(x-1) \rangle = 2p_0 p_1 + 2p_1(-p_0) + 2p_2 0 + 2p_3 0 = 0$$

The same reasoning for $\ell = -1$ gives us

$$\langle \psi(x), \phi(x-1) \rangle = 2p_0 0 + 2p_1 0 + 2p_2 p_3 + 2p_3(-p_2) = 0$$

The other ℓ all follow because the q_k's involved are all 0. Thus, if ψ has the form

$$\psi(x) = p_3\phi(2x) - p_2\phi(2x-1) + p_1\phi(2x-2) - p_0\phi(2x-3), \qquad (3)$$

then we know that it is in W_0. It is easy to show that if we translate $\psi(x)$ to $\psi(x-m)$, where $m \in \mathbb{Z}$, then $\psi(x-m)$ is also in W_0. We can say even more. Using (3) and the orthonormality of $\{\sqrt{2}\phi(2x-k)\}_{k\in\mathbb{Z}}$, with a little work one can show that $\{\psi(x-m)\}_{m\in\mathbb{Z}}$ is an orthonormal set for W_0. We leave doing this as an exercise. Showing that it is also a basis for W_0 is done in an appendix in the book.

We now return to the general case discussed in the text. Just as for the Haar wavelet, $\{\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for the j^{th} level wavelet space. We again have the decomposition

$$V_j = V_{j-1} \oplus W_{j-1}, \quad V_{j-1} \perp W_{j-1}$$

There is one other *very* important fact that we want to mention. The collection of all of functions $\{\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for the whole space of signals, $L^2(\mathbb{R})$.

We will discuss decomposition and reconstruction in class.