Midterm test – take-home part. This part is due on Thursday, 10/25/06. You may not get help on the test from anyone except your instructor.

1. **(15 pts.)** Prove a two-dimensional version of the Weierstrass Approximation Theorem.

2. **(10 pts.)** A Lebesgue measurable function $g : [0.1] \to \mathbb{R}$ is said to be *simple* if its range consists of finitely many values, $a_1 < a_2 < \cdots < a_n$. (For example, the characteristic function χ_A of a measurable set A is simple because its range is $\{0, 1\}$.) Let $E_j = g^{-1}\{a_j\}$. Show that

$$\int_0^1 g(t)dt = \sum_{j=1}^{n} a_j m(E_j).$$

3. Black and white images are stored as $m \times n$ matrices with integers coefficients – 0 to 255 (8-bit) is common. Each entry in A represents a gray scale pixel value. The location in the matrix represents the position of the pixel in the image. Thus a 600×800 image is represented by a matrix with 480,000 entries. The energy in the image is taken to be the sum of the squares of the entries: $E_A := \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k}^2$.

   (a) **(5 pts.)** Show that $E_A = \mathrm{Tr}(A^*A) = \mathrm{Tr}(AA^*)$. Here Tr is the trace. The quantity $\|A\|_F := \sqrt{E_A}$ is called the Frobenius norm of A. (See §1.5.1 in the text.)

   (b) **(5 pts.)** Let $A = U\Sigma V^T$ be the SVD for A, where $U = (u_1 \cdots u_m)$ and $V = (v_1 \cdots v_n)$ are orthogonal matrices. Show that $E_A = E_\Sigma = \sum_{j=1}^{r} \sigma_j^2$, where $r = \mathrm{rank}(A)$. That is, show that E_A is the sum of the squares of the singular values of A.

   (c) **(5 pts.)** Using the "thin" SVD, we can write $A = \sum_{j=1}^{r} \sigma_j u_j v_j^T$, and then we can compress an image by truncating this sum at $s < r$. The matrix $B = \sum_{j=1}^{s} \sigma_j u_j v_j^T$ then represents the compressed image. Show that % energy retained $= \frac{\sum_{j=1}^{s} \sigma_j^2}{\sum_{j=1}^{r} \sigma_j^2} \times 100\%$ and % compression $= \frac{s}{r} \times 100\%$,

   (d) **(10 pts.)** Find a black and white image and use the method above to compress the image for various values of s/r. How much compression can you achieve without sacrificing too much image quality? How much energy is retained?