1 Construction of Cubic Splines

The cubic splines are piecewise cubic polynomials on $[0,1]$. We denote the set of cubic splines by $S^h(3,1)$. The cubic splines can be used to interpolate simultaneously both pointwise values of a function and pointwise values of the derivatives on a set of knots $\{x_j\}_{j=1}^n$. That is, if the values $f(x_j)$ and $f'(x_j)$ are known, then there exists a cubic spline $s \in S^h(3,1)$ satisfies both $s(x_j) = f(x_j)$ and $s'(x_j) = f'(x_j)$. By a formula derived in the previous set of notes, the dimension of $S^{\frac{1}{n}}(3,1)$ is $2n+2$.

We construct a basis of functions for $S^{\frac{1}{n}}(3,1)$ by first constructing two interpolating functions. Consider the interval $[0,1]$ and the problem of finding a cubic polynomial $\phi(x)$ such that $\phi(0) = 1$, and $\phi(1) = \phi'(1) = \phi'(0) = 0$. Then, a polynomial of the form

$$\phi(x) = A(x-1)^3 + B(x-1)^2$$

satisfies $\phi(1) = \phi'(1) = 0$. Substituting the values for $\phi(0) = 1$ and $\phi'(0) = 0$ yields $-A + B = 1$ and $3A - 2B = 0$, which has the solution $A = 2$ and $B = 3$. Then, after re-arranging, we see that

$$\phi(x) = 2(x-1)^3 + 3(x-1)^2 = (x-1)^2(2x+1).$$

We then define

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{1}$$

The function ϕ yields zero derivative data at the endpoints, and is one at $x = 0$. The function ϕ will be used to interpolate the pointwise values of a function, while yielding zero derivative data on each of the knots.

We next construct a function ψ that takes zero value at the endpoints, but assumes a derivative value of one at 0. We let ψ be the cubic function

$$\psi(x) = A(x-1)^3 + B(x-1)^2$$

which already satisfies $\psi(1) = \psi'(1) = 0$. The condition $\psi(0) = 0$ implies $A = B$ and the condition $\psi'(0) = 1$ implies $3A - 2B = 1$. Combining these conditions yields the function

$$\psi(x) = x(x-1)^2.$$

We then define

$$\psi(x) = \begin{cases} x(|x|-1)^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{2}$$

We now construct a basis for $S^{\frac{1}{n}}(3,1)$ by using shifts and translates of the ϕ and ψ functions defined in (**??**) and (**??**). We define

$$\phi_j(x) := \phi(nx - j). \tag{3}$$

Notice that $\phi_0(x) = \phi(nx)$ and $\phi_j(x) = \phi(n(x - \frac{j}{n})) = \phi_0(x - \frac{j}{n})$. That is, $\phi_j(x)$ is $\phi_0(x)$ translated by $\frac{j}{n}$ and furthermore that $\phi_j(x)$ is supported on the interval $[\frac{j-1}{n}, \frac{j+1}{n}]$.

To construct the ψ_j basis functions, we first consider the derivative of $\psi(nx - j)$. We note that

$$\frac{d}{dx}\bigg|_{x=\frac{j}{n}} (\psi(nx - j)) = n\psi'(nx - j)\bigg|_{x=\frac{j}{n}} = n\psi'(0) = n.$$

From this computation, we see that it is not correct to choose $\psi_j(x) = \psi(nx - j)$. We must properly scale it by n. Consequently, we define

$$\psi_j(x) = \frac{1}{n}\psi(nx - j) \tag{4}$$

and we see the the support of ψ_j is also contained in the interval $[\frac{j-1}{n}, \frac{j+1}{n}]$.

2 Interpolation with Cubic Splines

We consider the problem of interpolating a function f at a set of knots spaced equally by distance $\frac{1}{n}$ using cubic splines constructed in the previous section. In addition to interpolating pointwise values of f at the knots, the cubic splines allow for interpolation of derivative data of f at the knots. Let $s \in S^{\frac{1}{n}}(3,1)$ be the interpolant

$$s(x) = \sum_{j=0}^{n} f(x_j)\phi_j(x) + f'(x_j)\psi_j(x).$$

It follows by the formulas from the previous section that $s(\frac{j}{n}) = f(\frac{j}{n})$ and $s'(\frac{j}{n}) = f'(\frac{j}{n})$. This demonstrates that the cubic splines can be used to simultaneously interpolate pointwise values of f and pointwise values of f'. We have not demonstrated that the set of functions $\{\phi_j, \psi_j\}_{j=0}^n$ are a basis for $S^{\frac{1}{n}}(3,1)$. We note that there are $n+1$ of each, which gives $2n+2$ total functions, which is the dimension of $S^{\frac{1}{n}}(3,1)$. Therefore, it suffices to show that the set $\{\phi_j, \psi_j\}_{j=0}^n$ are linearly independent. Suppose not. Then, there exists an $s(x) \in S^{\frac{1}{n}}(3,1)$ such that $s(\frac{j}{n}) = s'(\frac{j}{n}) = 0$ for $j = 0, \ldots, n$ but $s \neq 0$. Consider an interval $[\frac{j}{n}, \frac{j+1}{n}]$. On this interval, we know that s is a cubic polynomial of the form

$$s(x) = A(x - \frac{j}{n})^2(Cx + D)$$

since s and s' both have zeros at $\frac{j}{n}$. Alternatively, we may express $s(x) = A(x - \frac{j}{n})^3 + B(x - \frac{j}{n})^2$. Substituting $x = \frac{j+1}{n}$ yields

$$0 = \frac{A}{n^2} + \frac{B}{n} = 0$$

and

$$0 = \frac{3A}{n^2} + \frac{2B}{n} = 0$$

by using $s'(\frac{j+1}{n}) = 0$. This yields a solution of $A = B = 0$. Therefore, the set $\{\phi_j, \psi_j\}_{j=0}^n$ is linearly independent, and hence spans $S^{\frac{1}{n}}(3,1)$.

3 Finite Element Methods and Galerkin Methods

Consider the problem of finding the "smoothest" function in $S^{\frac{1}{n}}(3,1)$ such that at the knots x_j, $s(x_j) = f_j$ for $j = 0, \ldots, n$. To define "smoothest", we seek a function s that minimizes

$$\|s\|^2 := \int_0^1 (s''(x))^2 \, dx \tag{5}$$

over all $s \in S^{\frac{1}{n}}(3,1)$ in which $s(x_j) = f_j$ for $j = 0, \ldots, n$.

Since s is a piecewise cubic function, s'' exists and is piecewise continuous. Therefore, the equation (**??**) is well defined for all of $s \in S^{\frac{1}{n}}(3,1)$. In fact, it can be shown that (**??**) is an inner product on the set of functions in $S^{\frac{1}{n}}(3,1)$ which are zero at the endpoints.

Any function $s \in S^{\frac{1}{n}}(3,1)$ such that $s(x_j) = f_j$ can be written in the form

$$s(x_j) = \sum_{j=0}^{n} f_j \phi_j(x) - \sum_{j=0}^{n} \alpha_j \psi_j(x).$$

Let $f = \sum_{j=0}^{n} f_j \phi_j(x)$. We seek to find coefficients α that minimize the norm of s. That is, we want to solve the problem

$$\min_{g \in \text{span}(\psi_j)} \|f - g\|. \tag{6}$$

This is a least-squares problem which can be solved by solving the normal equations. We expand $g = \sum_{j=0}^{n} \alpha_j \psi_j$ and we seek to find coefficients α_j such that

$$\langle f - g, \psi_k = 0 \rangle \tag{7}$$

for $k = 1, \ldots, n$. Expanding g in terms of the ψ_k functions, we see this yields a system of equations

$$\sum_{j=0}^{n} \alpha_j \langle \psi_j, \psi_k \rangle = \langle f, \psi_k \rangle.$$

Due to the compact support of ψ_k, we see that

$$\langle \psi_j, \psi_k \rangle = \int_0^1 \psi_j''(x) \psi_k''(x)\, dx = \int_{[\frac{j-1}{n}, \frac{j+1}{n}] \cap [\frac{k-1}{n}, \frac{k+1}{n}]} \psi_j''(x) \psi_k''(x)\, dx. \tag{8}$$

This integral is nonzero only for $k = j-1$, $k = j$ or $k = j+1$. Therefore the matrix $G_{jk} = \langle \psi_j, \psi_k \rangle$, G is a tridiagonal matrix.