Bounded Operators

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November, 2013

1 Bounded operators & examples

Let V and W be Banach spaces. We say that a linear transformation $L : V \to W$ is *bounded* if and only there is a constant K such that $\|Lv\|_W \le K\|v\|_V$ for all $v \in V$. Equivalently, L is bounded whenever

$$\|L\|_{op} := \sup_{v \neq 0} \frac{\|Lv\|_W}{\|v\|_V} \tag{1.1}$$

is finite. $\|L\|_{op}$ is called the norm of L. Frequently, the same operator may map another space $\widetilde{V} \to \widetilde{W}$, rather than $V \to W$. When this happens, we will need to note which spaces are involved. For instance, if V and W are the spaces involved, we will use the notation $\|L\|_{V \to W}$ for the operator norm. In addition to the expression given in (1.1), it is easy to show that $\|L\|_{op}$ is also given by

$$\|L\|_{op} := \inf\{K > 0 \colon \|Lv\|_W \le K\|v\|_V \ \forall v \in V\}. \tag{1.2}$$

As usual, we say $L : V \to W$ is continuous at $v \in V$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Lu - Lv\|_W < \varepsilon$ whenever $\|u - v\|_V < \delta$. Of course, this is just the standard definition of continuity. Be aware that it holds whether or not L is linear. When L is linear, the distinction between u, v becomes irrelevant, because $\|Lu - Lv\|_W = \|L(u - v)\|_W$. From this it immediately follows that L will be continuous at every $v \in V$ whenever it is continuous at $v = 0$. The proposition below connects boundedness and continuity for linear transformations. The proof is left as an exercise.

Proposition 1. *A linear transformation $L : V \to W$ is continuous if and only if it is bounded.*

We will now provide a number of examples of bounded operators and unbounded operators.

Example 1. Let $L : C[0,1] \to C[0,1]$ be given by $Lu(x) = \int_0^1 k(x,y)u(y)dy$, where $k \in C(R)$, $R = [0,1]\times[0,1]$. We have that $|Lu(x)| \le \int_0^1 |k(x,y)|\,|u(y)|dy$, so $|Lu(x)| \le \|k\|_{C(R)}\|u\|_{C([0,1])}$. Consequently, $\|L\|_{C\to C} \le \|k\|_{C(R)}\|u\|_{C([0,1])}$

Example 2. Again let $R = [0,1] \times [0,1]$. If $k \in L^2(R)$, then k is called a *Hilbert-Schmidt kernel.* The linear operator $Lu(x) = \int_0^1 k(x,y)u(y)dy$ maps $L^2[0,1] \to L^2[0,1]$ and is bounded. Moreover, $\|L\|_{L^2\to L^2} \le \|k\|_{L^2(R)}$.

Proof. We will first show that $k(x,y)u(y) \in L^1(R)$. To see this, observe that, by Schwarz's inequality,

$$\begin{aligned}\int_0^1\int_0^1 |k(x,y)u(y)|dxdy &\le \|k\|_{L^2(R)}\bigg(\int_0^1\int_0^1 |u(y)|2|^2dxdy\bigg)^{1/2}\\ &\le \|k\|_{L^2(R)}\|u\|_{L^2[0,1]}.\end{aligned}$$

Consequently, $k(x,y)u(y) \in L^1(R)$. By the Fubini-Tonelli Theorem, we have that, as function of x, $\int_0^1 k(x,y)u(y)dy \in L^1[0,1]$ and, as function of y, $\int_0^1 k(x,y)u(y)dx \in L^1[0,1]$. In any event, $Lu(x)$ is in $L^1[0,1]$. Next, because $k \in L^2(R)$, the integral $\int_0^1\int_0^1 |k(x,y)|^2dxdy = \|k\|^2_{L^2(R)}$ is finite. Thus $|k(x,y)|^2 \in L^1([0,1])$. The Fubini-Tonelli Theorem then implies that for almost every x, $|k(x,y)|^2 \in L^1[0,1]$. Thus, applying Schwarz's inequality yields

$$|Lu(x)|^2 \le \bigg|\int_0^1 |k(x,y)|u(y)|dy\bigg|^2 \le \|u\|^2_{L^2[0,1]}\int_0^1 |k(x,y)|^2dy.$$

Another application of the Fubini-Tonelli Theorem implies that the right side above is in $L^1[0,1]$. Integrating this in x yields

$$\int_0^1 Lu(x)|^2dx \le \|u\|^2_{L^2[0,1]}\int_0^1 |k(x,y)|^2dydx = \|k\|^2_{L^2(R)}\|u\|^2_{L^2[0,1]}.$$

It follows that $Lu \in L^2[0,1]$ and $\|L\|_{L^2\to L^2} \le \|k\|_{L^2(R)}$. □

Example 3. Let $\mathcal{H} = L^2[0,1]$. The differentiation operator $D = \frac{d}{dx}$ is defined on all $f \in C^1[0,1]$, which is dense in $\mathcal{H}$ because it contains the set of polynomials. The question is whether D is bounded, or at least can be extended to a bounded operator on $\mathcal{H}$. The answer is no. Let $u_n(x) :=$

$\sqrt{2}\sin(n\pi x)$. These functions are in $C^1[0,1]$ and they satisfy $\|u_n\|_{\mathcal{H}} = 1$. Since $Du_n = n\pi\sqrt{2}\cos(n\pi x)$, $\|Du_n\|_{\mathcal{H}} = n\pi$. Consequently,

$$\frac{\|Du_n\|_{\mathcal{H}}}{\|u_n\|_{\mathcal{H}}} = n\pi \to \infty, \text{ as } n \to \infty.$$

Thus D is an *unbounded* operator.

The situation changes if we use a different space. Consider the Sobolev space $H^1[0,2\pi] := \{f \in L^2[0,2\pi] \colon f' \in L^2[0,2\pi]$, where f' is computed in a distributional sense. $H^1[0,2\pi]$ has the inner product $\langle f,g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$. The operator $D : H^1 \to L^2$ turns out to be bounded. In fact, $\|D\|_{H^1\to L^2} = 1$.

2 Closed subspaces

The usual definition of subspace holds for Banach spaces and for Hilbert spaces. Such subspaces inherit norms and/or inner products from the larger spaces. They are said to be *closed* if they contain all of their limit points.

Finite dimensional subspaces are always closed. Earlier, when we discussed completeness of an orthonormal set $U = \{u_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, we saw that the space $\mathcal{H}_U = \{f \in \mathcal{H} \colon f = \sum_n \langle f, u_n\rangle u_n\}$ is closed in $\mathcal{H}$. When $C[0,1]$ is considered to be a subspace of $L^2[0,1]$, it is not closed. However, $C[0,1]$ *is* a closed subspace of $L_\infty[0,1]$.

Given a subspace V of a Hilbert space $\mathcal{H}$, we define the *orthogonal complement* of V to be

$$V^\perp := \{f \in \mathcal{H} \colon \langle f, g\rangle \ \forall g \in V\}.$$

Proposition 2. *$V^\perp$ is a closed subspace of $\mathcal{H}$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $V^\perp$ that converges to a function $f \in \mathcal{H}$. Since each f_n is in $V^\perp$, $\langle f_n, g\rangle = 0$ for every $g \in V$. Also, because the inner product is continuous, $\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle$. It immediately follows that $\langle f, g\rangle = 0$. so $f \in V^\perp$. Consequently, $V^\perp$ is closed in $\mathcal{H}$. □

Bounded linear operators mapping $V \to W$, where V and W are Banach spaces, have all of the usual subspaces associated with them. Let $L : V \to W$ be bounded and linear. The domain of L is $D(L) = V$. The range of L is defined as $R(L) := \{w \in W \colon \exists v \in W$ for which $Lv = W\}$. Finally, the null space (or kernel) of L is $N(L) := \{v \in V \colon Lv = 0\}$.

Proposition 3. *If $L : V \to W$ be bounded and linear, then the null space $N(L)$ is a closed subspace of V.*

Proof. The proof again relies on the continuity of L. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $N(L)$ that converges to $f \in V$. By Proposition 1, L is continuous, so $\lim_{n\to\infty} Lf_n = Lf$. But, because $f_n \in N(L)$, $Lf_n = 0$. Combining this with $\lim_{n\to\infty} Lf_n = Lf$, we see that $Lf = 0$ and so $f \in N(L)$. Thus, $N(L)$ is a closed subspace of V. $\square$