The Closed Range Theorem

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November, 2013

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and the compact operators on $\mathcal{H}$, respectively.

Our aim here is to prove that the range of an operator of the form $L = I - \lambda K$, where K is compact, is closed. We will begin with this lemma.

Lemma 0.1. *Let $K \in \mathcal{C}(\mathcal{H})$, $\lambda \in \mathbb{C}$, and $L = I - \lambda K$. If $f \in \overline{R(L^*)}$, then there is a constant $c > 0$, independent of f, such that $\|Lf\| \geq c\|f\|$.*

Proof. If not, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \overline{R(L^*)}$ such that $\|f_n\| = 1$ and $\|Lf_n\| \to 0$ as $n \to \infty$. Note that $Lf_n = f_n - \lambda K f_n$, so $f_n = \lambda K f_n + Lf_n$. Since the f_n's are bounded and K is compact, we may choose a subsequence $\{f_{n_k}\}$ such that $\{Kf_{n_k}\}$ is convergent. Thus, $\lim_{k\to\infty} f_{n_k} = \lambda \lim_{k\to\infty} Kf_{n_k} + \lim_{k\to\infty} Lf_{n_k}$. Both terms on the right are convergent, so f_{n_k} is also convergent. Let $f = \lim_{k\to\infty} f_{n_k}$. By the previous equation, we have that $f = \lambda K f$, so $Lf = 0$ – i.e., $f \in N(L)$. In addition, because $\overline{R(L^*)}$ is closed, $f \in \overline{R(L^*)}$. Since these spaces are orthogonal, f is orthogonal to itself and, consequently, $f = 0$. However, $\lim_{n\to\infty} \|f_n\| = 1 = \|f\|$. This is a contradiction. □

Theorem 0.2 (Closed Range Theorem). *If $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then the range of the operator $L = I - \lambda K$ is closed.*

Proof. We want to show that if there is sequence $\{g_n\} \subset R(L)$ such that $g_n \to g$, then $g = Lf$ for some $f \in \mathcal{H}$. To begin, note that the solution f_n to $g_n = Lf_n$ is not unique if $N(L) \neq \{0\}$. Since $\mathcal{H} = N(L) \oplus \overline{R(L^*)}$, with the two spaces being orthogonal, we may make a unique choice by requiring that f_n be in $\overline{R(L^*)}$. Lemma 0.1 then implies that $\|g_n - g_m\| = L(f_n - f_m)\| \geq c\|f_n - f_m\|$. Because the convergent sequence $\{g_n\}$ is Cauchy, this inequality also implies that $\{f_n\}$ is Cauchy. Thus, $\{f_n\}$ is convergent to some $f \in \mathcal{H}$. It follows that $g = \lim_{n\to\infty} Lf_n = Lf$, so $g \in R(L)$. □

The Closed Range Theorem allows us to apply the Fredholm alternative to equations of the form $u - \lambda K u = f$. Thus, we have the following result, which applies, for example, to finite rank or Hilbert-Schmidt operators.

Corollary 0.3. *Let $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. The equation $u - \lambda K u = f$ has a solution if and only if $f \in N(I - \bar{\lambda} K^*)^{\perp}$.*