A Resolvent Example

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Problem. Let $k(x,y) = x^2y$, $Ku(x) = \int_0^1 k(x,y)u(y)dy$, and $Lu = u - \lambda Ku$. Assume that L has closed range.

1. Determine the values of λ for which $Lu = f$ has a solution for all f. Solve $Lu = f$ for these values of λ.
2. For the remaining values of λ, find a condition on f the guarantees a solution to $Lu = f$. When f satisfies this condition, solve $Lu = f$.

Solution. (1) Because $R(L)$ is closed, the Fredholm alternative applies. We begin by finding $N(L^*)$. First, we have that $L^* = I - \bar{\lambda}K^*$, where $K^*v = \int_0^1 k(y,x)v(y)dy = \int_0^1 y^2xv(y)dy$. We want to find all v for which $L^*v = v - \bar{\lambda}\int_0^1 y^2xv(y)dy = 0$. Note that $v = \bar{\lambda}x\int_0^1 y^2v(y)dy$, so $v = Cx$. Putting this back into the equation for v yields $Cx = \bar{\lambda}Cx\int_0^1 y^2ydy = C(\bar{\lambda}/4)x$. Thus, $C = (\bar{\lambda}/4)C$. If $\bar{\lambda}/4 \neq 1$, then $C = 0$ and $N(L^*) = \{0\}$. Thus, if $\bar{\lambda}/4 \neq 1$ – i.e., $\lambda \neq 4$, $Lu = f$ has a solution for all $f \in L^2[0,1]$. To find u, note that $u - \lambda x^2\int_0^1 yu(y)dy = f$, and so we only need to find $\int_0^1 yu(y)dy$. The trick for doing this is to multiply $Lu = f$ by x and then integrate. Doing this results in $\int_0^1 yu(y)dy - \frac{\lambda}{4}\int_0^1 yu(y)dy = \int_0^1 yf(y)dy$. From this we get $\int_0^1 yu(y)dy = \frac{1}{1-\lambda/4}\int_0^1 yf(y)dy$. Finally, we arrive at

$$u(x) = f(x) + \frac{4\lambda}{4-\lambda}x^2\int_0^1 yf(y)dy = u(x) + \frac{4\lambda}{4-\lambda}Kf(x).$$

In operator form,

$$(I - \lambda K)^{-1} = I + \frac{4\lambda}{4-\lambda}K$$

The operator $(I - \lambda K)^{-1}$ is called the *resolvent* of K.

(2) When $\lambda = 4$, $N(L^*) = \text{span}\{x\}$. By the Fredholm alternative, $Lu = f$ has a solution if and only if $\int_0^1 xf(x)dx = 0$. To solve $u - 4x^2\int_0^1 yu(y)dy = f$ for u, we first note that $\int_0^1 yu(y)dy$ is *not* determined. This is because $\int_0^1 yu(y)dy - \frac{4}{4}\int_0^1 yu(y)dy = \int_0^1 yf(y)dy = 0$. This really only says that $0 = 0$; we only have consistency. The constant $C = \int_0^1 yu(y)dy$ is thus arbitrary. The solution we arrive at has the form $u(x) = f(x) + Cx^2$.