Math 641
Dec. 3, 2013

§1

Last time: spectral theory — Today: (1) Example. (2) Intr. to distrib.

Example. Consider the e-val. problem

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u(1) + u'(1) = 0.$$

Show that there ~~are~~ are infinitely many e-vals & that the corresp. e-fncts. are complete. Find an eqn. for the e-vals.

~~Soln. Look at $u'' = -\lambda u$, the~~

Soln. Consider $u'' = f(x)$, w/ $u(0) = 0$ etc. We want to put this in the form of an integr. eqn. To do this, we will find a "Green's" fnct, $G(x, y)$, s.t.

$$u = \int_0^1 G(x,y) f(y)\,dy.$$

w/o distributions, this can be done via var. of param. That is, we write $u(x) = c_1(x) u_1(x) + c_2(x) u_2(x)$, where u_1, u_2 are homogeneous solns. It is best to take

$$u_1(0) = 0 \quad \& \quad u_2'(1) + u_2(1) = 0.$$

For u_1: $u_1(x) = x$; for $u_2(x)$, $u_2(x) = x - 2$. Also,

$$c_1' u_1 + c_2' u_2 = 0, \quad c_1' u_1' + c_2' u_2' = f(x).$$

$$\begin{pmatrix} x & x-2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \Rightarrow \begin{array}{l} c_1' = -\frac{1}{2} f(x)(x-2) \\ c_2' = \frac{1}{2} f(x)\, x \end{array}$$

I.C. $c_2(0) = 0$, $c_1(1) = 0$.

$$\Rightarrow c_1 = \frac{1}{2}\int_x^1 (y-2) f(y)\,dy, \quad c_2 = \frac{1}{2}\int_0^x f(y)\, y\,dy.$$

$$\Rightarrow u(x) = \frac{1}{2}\int_x^1 x(y-2)f(y)\,dy + \frac{1}{2}\int_0^x y(x-2)f(y)\,dy$$

$$\Rightarrow u(x) = \int_0^1 G(x,y)f(y)\,dy, \quad \text{where —}$$

$$G(x,y) = \begin{cases} \frac{1}{2}x(y-2), & x \le y, \\ \frac{1}{2}y(x-2), & y \le x. \end{cases}$$

G is called the Green's function for the problem.

Define the operator $Gf := \int_0^1 G(x,y)f(y)\,dy$. By our construction of $G(x,y)$, if $u = Gf$, then $u'' = f$, $u(0) = 0$, $u(1) + u'(1) = 0$. With a little bit of work, we can show that this ~~holds~~ ~~for all~~ holds for all $f \in L^2$, provided the derivatives are ~~interpreted~~ interpreted in the right way. Of course, if f is continuous, then $u'' = f$ in the usual sense.

~~One can see~~ It is easy to show that the kernel ~~G(x,y) is G(x,y)~~ $G(x,y)$ satisfies $G(x,y) = G(y,x)$ and $G(x,y) \in L^2([0,1]\times[0,1])$, thus, $G(x,y)$ is a symmetric Hilbert-Schmidt kernel. This implies that $G = G^*$ and that G is compact.

We now turn to the eigenvalue problem that we started with; namely,

$$\cancel{u'' + \lambda u,}$$

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u'(1) + u(1) = 0.$$

First, note that $\lambda \neq 0$. If $\lambda = 0$, then $u'' = 0$. Consequently, $u = Ax + B$. By $u(0) = 0$, we have $B = 0$. By $u(1) + u'(1) = 0$, we have $A + B + A = 0$. Since $B = 0$, $A = 0$. Thus, $u \equiv 0$ and $\lambda = 0$ is not an eigenvalue. Second, we also have

$$u = -\lambda G u, \text{ so } Gu = -\frac{1}{\lambda} u.$$

Thus, $-\frac{1}{\lambda}$ is an eigenvalue of G. In fact, every nonzero eigenvalue of G satisfies

$$Gu = \mu u$$

$$\Rightarrow \quad u'' + \frac{1}{\mu} u = 0, \quad u(0) = 0, \quad u(1) + u'(1) = 0.$$

That is, $\lambda = \frac{-1}{\mu}$ is an eigenvalue of the original problem. ~~Third, G has no 0 eigenv~~ Third, 0 is not an eigenvalue of G. (We will show this later.)

The point is that the ~~first~~ first and second items above imply that the nonzero eigenvalues of both problems have the same set of eigenvectors.

This set can be chosen to be o.n. Moreover, by the spectral theory of compact operators, the eigenvectors of G, including those for 0, form a complete set. Since 0 is <u>not</u> an eigenvalue of G, the eigenvectors for the nonzero eigenvalues form a complete set. These eigenvectors are the same for both problems. Thus, the eigenvectors of $u''+\lambda u=0$, $u(0)=0$, $u'(1)+u(1)=0$ ~~for~~ form a complete o.n. set.

We briefly sketch solving the eigenvalue problem ~~set~~ itself. It is easy to show that all of the eigenvalues in the problem are positive. Thus the ~~eigenvalues~~ eigenvectors (~~eigenfunction~~ have the form

$$u = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

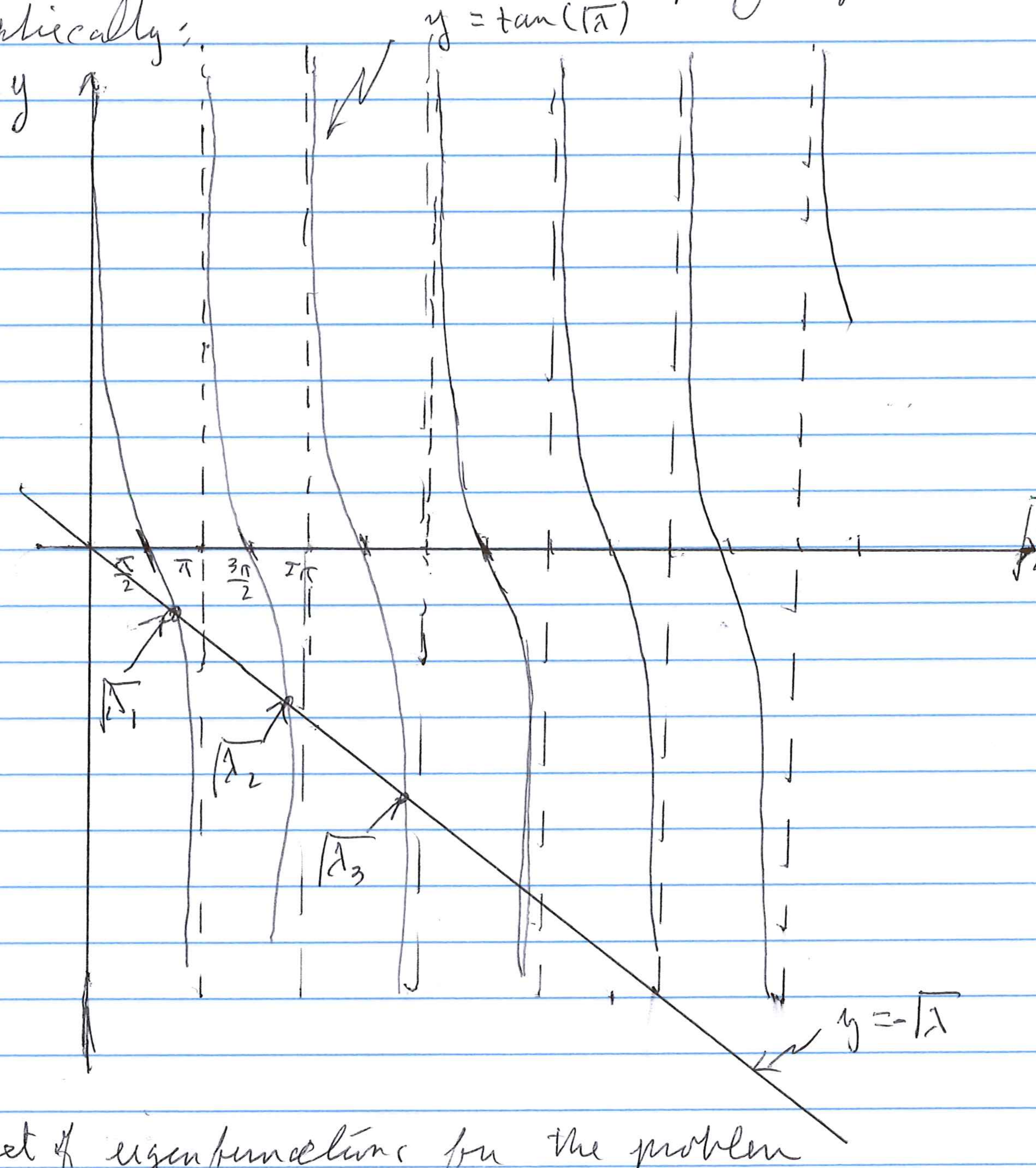
Since $u(0)=0$, we have $B=0$. Now, A cannot be ~~zer~~ zero, because an eigenvector is never 0. Thus, we only need to look at $u=\sin(\sqrt{\lambda}x)$. At $x=1$, this function satisfies the boundary condition

$$0 = u(1)+u'(1) = \sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda}).$$

That is,

$$\tan(\sqrt{\lambda}) = -\sqrt{\lambda}.$$

This is a transcendental equation for $\sqrt{\lambda}$. It can only be solved numerically. The solutions can however be displayed ~~graphically~~ graphically:



The set of eigenfunctions for the problem is then $\{\sin(\sqrt{\lambda_n}\, x)\}_{n=1}^{\infty}$, where $\sqrt{\lambda_n}$ is illustrated in the graph above.

We want to show that G doesn't have a zero eigenvalue. There are two ways to do this. Suppose that there is a $u \in \mathcal{H} = L^2$ such that $Gu = 0$. The first way is by differentiating in the appropriate sense:

$$u'' = (Gu)'' = 0,$$

but also $(Gu)'' = u$. Thus, $u = 0$. Another way that avoids distributional derivatives is to note that for all $f \in C[0,1]$, $v = Gf$ does satisfy $v'' = f$, $v(0) = 0$, $v'(1) + v(1) = 0$. If $Gu = 0$, then, since $G = G^*$,

$$0 = \langle Gu, f \rangle = \langle u, Gf \rangle = \langle u, v \rangle.$$

However, one can show ~~that~~ that $v = Gf$ form a dense set in L^2. (Why?) Thus, ~~for all~~ $\langle u, v \rangle = 0 \Rightarrow \langle u, g \rangle = 0$ for all $g \in L^2$. It follows that $u = 0$, and consequently that 0 is not an eigenvalue of G. This means that

$$\cancel{\left\{ \sin\left(\sqrt{\lambda_n}\, x\right) \right\}_{n=1}}$$

$$\left\{ \sin\left(\sqrt{\lambda_n}\, x\right) \right\}_{n=1}^{\infty}$$

is complete.

Green's Function + Distributions Go back to our example —

$$u(x) = \int_0^1 G(x,y)\,f(y)\,dy, \quad u'' = f, \; u(0)=0, \; u'(1)+u(1)=0$$

It is easy to show that u satisfies the b.c. Let's "pretend" we can diff. under the integ. sign —

$$u''(x) = \int_0^1 \underbrace{\frac{\partial^2 G}{\partial x^2}(x,y)}\, f(y)\,dy = f(x).$$

Dirac δ (or pt. eval. fnct'l). $\Rightarrow \frac{\partial^2 G}{\partial x^2}(x,y) = \delta(x-y)$.

~~(1) W want: (1)~~

want: $\frac{\partial^2 G}{\partial x^2} = 0, \quad x \neq y, \quad 0 < x < 1.$

$$\int_{y-\varepsilon}^{y+\varepsilon} \frac{\partial^2 G}{\partial x^2}(x,y)\,dy = \frac{\partial G}{\partial x}(y+\varepsilon, y) - \frac{\partial G}{\partial x}(y-\varepsilon, y) = 1$$

Let $\varepsilon \downarrow 0$, $\frac{\partial G}{\partial x}(y^+,y) - \frac{\partial G}{\partial x}(y^-,y) = 1$ ← Jump in $\frac{\partial G}{\partial x}$ at $x=y$.

Continuity. $G(y^+,y) = G(y^-,y)$.

For our case — $\frac{\partial^2 G}{\partial x^2} = 0$, $x \neq y$. $\Rightarrow G(x,y) = Ax+B$, $x<y$ & $G(x,y) = Cx+D$, $x>y$. For $x<y$, $G(0,y) = 0 = B$

$\therefore G(x,y) = Bx$, $0<x<y$. At $x=1$, $\frac{\partial G}{\partial x}(1,y) + G(1,y) = 0$

$\Rightarrow C + C + D = 0 \Rightarrow D = -2C$.

$\therefore G(x,y) = C(x-2)$, $y<x$.

Using $G(y^+;y) = G(y^-;y)$, we get

(1) $C(y-2) = By$,

<u>Jump.</u> $\frac{\partial G}{\partial x}(y^+;y) - \frac{\partial G}{\partial x}(y^-;y) = 1$

$\Rightarrow$ (2) ~~C - B = 1~~, $C - B = 1$

Solving for A and C yields $C = \frac{y}{2}$ and ~~B=~~

$B = \frac{1}{2}(y-2)$. The Green's function thus has the form

$$G(x,y) = \begin{cases} \frac{x}{2}(y-2), & x \le y, \\ \frac{y}{2}(x-2), & y \le x. \end{cases}$$

This agrees with what we got before.

<u>Procedure.</u> Let L be a differential operator, subject to boundary conditions of some sort. To find the Green's function, we do this:

(1) $L_x G(x,y) = \delta(x,y)$. Thus, for $x \neq y$, $L_x G = 0$. Solve for G, with G satisfying the left boundary condition ($x < y$) and then for G when it satisfies the right boundary condition.

(2) <u>Jump.</u> $\int_{y^-}^{y^+} L_x G \, dx = 1$.

(3) <u>Continuity</u> $G(y^+,y) = G(y^-,y)$.

Exercise. Show that the Green's function for

$$Lu = \frac{d}{dx}\left((1+x)\frac{du}{dx}\right), \quad u(0)=0, \; u'(0)=0$$

is

$$G(x,y) = \begin{cases} -\log(1+x), & x \le y \\ -\log(1+y), & y \le x. \end{cases}$$