Banach Spaces and Hilbert Spaces*

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October 18, 2023

1 Complete Spaces

We will first discuss Banach spaces, since much of what we say applies to Hilbert spaces, without change.

Let V be a normed linear space over either the real or complex numbers. A sequence of vectors $\{v_j\}_{j=1}^{\infty}$ is a map from the natural numbers $\mathbb{N}$ to V. We say that v_j *converges* to $v \in V$ if

$$\lim_{j\to\infty} \|v_j - v\| = 0.$$

A sequence $\{v_j\}$ is said to be *Cauchy* if for each $\epsilon > 0$, there exists a natural number N such that $\|v_j - v_k\| < \epsilon$ for all $j, k \geq N$. Every convergent sequence is Cauchy, but there are many examples of normed linear spaces V for which there exist non-convergent Cauchy sequences. One such example is the set of rational numbers $\mathbb{Q}$. The sequence $(1.4, 1.41, 1.414, \dots)$ converges to $\sqrt{2}$ which is not a rational number. We say a normed linear space is *complete* if every Cauchy sequence is convergent in the space. The real numbers are an example of a complete normed linear space.

We say that a normed linear space is a Banach space if it is complete. We call a complete inner product space a Hilbert space. Consider the following examples:

1. Every finite dimensional normed linear space is a Banach space. Likewise, every finite dimensional inner product space is a Hilbert space.

*Revised 9/8/2020

2. Let $x = (x_1, x_2, ..., x_n, ...)$ be a sequence. The following spaces of sequences are Banach spaces:

$$\ell^p = \{x : \{\sum_{j=1}^{\infty} |x_j|^p\}^{1/p} = \|x\|_{\ell^p} < \infty\}, 1 \le p < \infty \tag{1.1}$$

$$\ell^\infty = \{x : \sup_j |x_j| < \infty\},\ \|x\|_{\ell^\infty} = \sup_j |x_j| \tag{1.2}$$

$$c = \{x : \lim_{j\to\infty} x_j \text{ exists}\}, \|x\|_c = \|x\|_\infty \tag{1.3}$$

$$c_0 = \{x : \lim_{j\to\infty} x_j = 0\}, \|x\|_{c_0} = \|x\|_\infty \tag{1.4}$$

Except for ℓ^∞. the spaces above are separable – i.e., each has a countably dense subset.

3. The following function spaces are Banach spaces:

$$C[0,1],\ \|f\|_{C[0,1]} = \max_{x\in[0,1]} |f(x)| \tag{1.5}$$

$$C^{(k)}[a,b],\ \|f\|_{C^{(k)}[a,b]} = \sum_{j=0}^{k} \sup_{x\in[0,1]} |f^j(x)| \tag{1.6}$$

$$L^p(I),\ \|f\| = \left(\int_I |f(x)|^p\right)^{\frac{1}{p}} \tag{1.7}$$

$$L^\infty(I),\ \|f\|_{L^\infty(I)} = \text{ess-sup}_{x\in I}|f(x)| \tag{1.8}$$

There are two Hilbert spaces among the spaces listed: the sequence space ℓ^2 and the function space L^2. In the result below, we will show that ℓ^∞ is complete. After that, we will show that $C[0,1]$ is complete, relative to the *sup-norm*, $\|f\|_{C[0,1]} = \max |f(x)|$. Of course, this means that both of them are Banach spaces.

Proposition 1.1. *The space ℓ^∞ is a Banach space.*

Proof. The norm on ℓ^∞ is given by[1] $\|x\|_\infty = \sup_j |x(j)|$. Let $\{x_n\}_{n=1}^\infty \subset \ell^\infty$ denote a Cauchy sequence of elements in ℓ^∞. We will show that this sequence converges to $x \in \ell^\infty$. Since $\{x_n\}$ is Cauchy, for each $\epsilon > 0$, there exists an N such that for all $n, m \ge N$,

$$\|x_n - x_m\|_\infty < \epsilon. \tag{1.9}$$

[1] To avoid double subscripts, we will use $x(j)$ instead od x_j to denote the j^{th} entry in a sequence.

This implies that $|x_n(j) - x_m(j)| < \epsilon$ for all j. Consequently, the sequence $\{x_n(j)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, and hence converges to some value $x(j)$. That is, $\lim_{n\to\infty} x_n(j) = x(j)$ exists. From (1.9), if we choose $\epsilon = 1$, then for all $n, m \geq N$, we have

$$\|x_n - x_m\|_\infty < 1.$$

In particular, it follows that

$$\|x_n\| < 1 + \|x_m\|, \qquad n, m \geq N.$$

Fix m. Then, for all $n \geq N$, $|x_n(j)| \leq \|x_n\|_\infty < 1 + \|x_m\|_\infty$. Letting $n \to \infty$, we see that

$$|x(j)| \leq 1 + \|x_m\|_\infty$$

holds uniformly in j. Therefore, $x \in \ell^\infty$. To complete the proof, we need to show that x_n converges to x in norm. We have

$$|x_n(j) - x_m(j)| < \epsilon \qquad \forall j \in \mathbb{N} \text{ and } n, m \geq N.$$

Let $n \to \infty$. Then, $x_n(j) \to x(j)$ so we have that

$$|x(j) - x_m(j)| < \epsilon \qquad \forall j \in \mathbb{N}, m \geq N.$$

Since this holds for all j, it follows that $\|x - x_m\|_\infty < \epsilon$ for all $m \geq N$. Therefore, the sequence x_m converges to $x \in \ell^\infty$. ∎

2 Continuous Functions

As we said above, the space of contiuous functions $C[0,1]$ is complete, relative to the sup norm. We will prove this below. After doing so, we will then show that $C[0,1]$ is not complete in $L^1[0,1]$. The point is that a space may be complete relative to one norm, but not in some other norm.

Proposition 2.1. *Relative to the sup norm, $C[0,1]$ is complete and is thus a Banach space.*

Proof. Let $\{f_n(t)\}_{n=1}^{\infty}$ be a Cauchy sequence in $C[0,1]$. Then, for every $\epsilon > 0$, there exists an N such that $\|f_n - f_m\| < \epsilon$ for all $n, m \geq N$. For any fixed $t \in [0,1]$, this implies that

$$|f_n(t) - f_m(t)| < \epsilon \qquad \forall\, m, n \geq N. \tag{2.1}$$

Thus, for t fixed, $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, and so it converges. Define $f(t)$ by the pointwise limit of this sequence:

$$f(t) = \lim_{n\to\infty} f_n(t), \ t \in [0,1]. \tag{2.2}$$

By taking the limit as $m \to \infty$ in (2.1), we see that

$$|f_n(t) - f(t)| \le \epsilon \qquad \forall\, n \ge N,$$

which holds *uniformly* for all $t \in [0,1]$. Consequently,

$$\|f_n - f\| = \sup_{t\in[0,1]} |f_n(t) - f(t)| \le \epsilon, \qquad \forall\, n \ge N, \tag{2.3}$$

and so $\lim_{n\to\infty} \|f_n - f\| = 0$.

What remains is to show that $f \in C[0,1]$. To do that, fix t, h so that $t, t+h \in [0,1]$. Let $\epsilon > 0$. By the triangle inequality,

$$|f(t+h) - f(t)| \le |f(t+h) - f_n(t+h)| + |f_n(t+h) - f_n(t)| + |f_n(t) - f(t)|.$$

Next, choose n so large that (2.3) holds for $\varepsilon/3$. It follows that

$$|f(t+h) - f(t)| \le \varepsilon/3 + |f_n(t+h) - f_n(t)| + \varepsilon/3..$$

Fix n. Because $f_n \in C[0,1]$, there exists a $\delta > 0$ such that

$$|f_n(t+h) - f_n(t)| < \frac{\epsilon}{3} \qquad \forall\, |h| < \delta.$$

Then this and the previous inequaity imply that

$$|f(t+h) - f(t)| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \varepsilon,$$

so $f \in C[0,1]$. Hence, $C[0,1]$ is complete. ∎

Just because a space is complete relative to one norm does not mean that the same space will also be complete in another. The example below illustrates this for the space of continuous functions, $C[0,1]$.

Example 2.2. *If we replace the sup norm on $C[0,1]$ with the L^1 norm, then $C[0,1]$ is not complete.*

Proof. To simplify the discussion, we will work with $[-1,1]$ rather than $[0,1]$. Consider the sequence of continuous functions $f_n \in C[-1,1]$ defined piecewise by

$$f_n(t) = \begin{cases} -1 & t \in [-1, -\frac{1}{n}] \\ nt & t \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & t \in [\frac{1}{n}, 1]. \end{cases}$$

Let $n > m$. We have

$$f_n(t) - f_m(t) = \begin{cases} -1 + mt & t \in [-\frac{1}{m}, -\frac{1}{n}] \\ (n-m)t & t \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 - mt & t \in [\frac{1}{n}, \frac{1}{m}] \\ 0 & t \in [-1, -\frac{1}{m}] \cup [\frac{1}{m}, 1]. \end{cases}$$

Because this function is odd, its absolute value is even. Computing the integrals over $[0,1]$ and doubling up the result yields

$$\|f_n - f_m\|_{L^1[-1,1]} = \frac{1}{m} - \frac{1}{n}.$$

Let $N > 2/\varepsilon$. Then, for $m, n \geq N$, we find that

$$\|f_n - f_m\|_{L^1[-1,1]} < \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so the sequence is Cauchy in the $L^1[-1,1]$ norm. In fact, the sequence is actually convergent in L^1 to its pointwise limit $f(t) = \lim_{n\to\infty} f_n(t)$, which is

$$f(t) = \begin{cases} -1 & t \in [-1, 0) \\ 0 & t = 0 \\ 1 & t \in (0, 1]. \end{cases}$$

We leave it as an exercise to show that $\|f - f_n\|_{L^1[-1,1]} < \frac{2}{n}$ for n large, so $\lim_{n\to\infty} \|f - f_n\|_{L^1} = 0$. However, the pointwise limit being discontinuous implies that $C[-1,1]$ is *not* a Banach space if the L^1 norm is used. ∎

Previous: adjoints and self-adjoint operators
Next: Lebesgue integration