Compact Sets and Compact Operators

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Throughout these notes, $\mathcal{H}$ denotes a separable Hilbert space. We will use the notation $\mathcal{B}(\mathcal{H})$ to denote the set of bounded linear operators on $\mathcal{H}$. We also note that $\mathcal{B}(\mathcal{H})$ is a Banach space, under the usual operator norm.

1 Compact and Precompact Subsets of $\mathcal{H}$

Definition 1.1. *A subset S of $\mathcal{H}$ is said to be* compact *if and only if it is closed and every sequence in S has a convergent subsequence. S is said to be precompact if its closure is compact.*

Proposition 1.2. *Here are some important properties of compact sets.*

1. Every compact set is bounded.
2. Let S be a bounded set. Then S is precompact if and only if every sequence has a convergent subsequence.
3. Let $\mathcal{H}$ be finite dimensional. Every closed, bounded subset of $\mathcal{H}$ is compact.
4. In an infinite dimensional space, closed and bounded is not enough.

Proof. Properties 2 and 3 are left to the reader. For property 1, assume that S is an unbounded compact set. Since S is unbounded, we may select a sequence $\{v_n\}_{n=1}^{\infty}$ from S such that $\|v_n\| \to \infty$ as $n \to \infty$. Since S is compact, this sequence will have a convergent subsequence, say $\{v_{n_k}\}_{k=1}^{\infty}$, which still will be unbounded.(Why?) Let $v = \lim_{k\to\infty} v_{n_k}$. Thus, for $\varepsilon = 1$ there is a positive integer K for which $\|v - v_{n_k}\| < 1$ for all $k \ge K$. By the triangle inequality, $\|v_{n_k}\| \le \|v\| + 1$. Now, the right side is bounded, but the left side isn't, since $\|v_{n_k}\| \to \infty$ as $k \to \infty$. This is a contradiction, so S must be bounded. For property 4, let $S = \{f \in \mathcal{H} \colon \|f\| \le 1\}$. Every o.n. basis $\{\phi_n\}_{n=1}^{\infty}$ is in S. However, for such a basis $\|\phi_m - \phi_n\| = \sqrt{2}$, $n \ne m$. Again, this means there are no Cauchy subsequences in $\{\phi_n\}_{n=1}^{\infty}$, and consequently, no convergent subsequences. Thus, S is not compact. $\square$

2 Compact Operators

Definition 2.1. *Let $K : \mathcal{H} \to \mathcal{H}$ be linear. K is said to be compact if and only if K maps bounded sets into precompact sets. Equivalently, K is compact if and only if for every bounded sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ the sequence $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$.*

Proposition 2.2. *If $K \in \mathcal{C}(\mathcal{H})$, then K is bounded – i.e., $\mathcal{C}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. In addition, $\mathcal{C}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$.*

Proof. We leave this as an exercise for the reader. □

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of a bounded operator K is finite dimensional, then we say that K is a finite-rank operator.

Proposition 2.3. *Every finite-rank operator K is compact.*

Proof. The range of K is finite dimensional, so every bounded subset of the range is precompact. Let $S \subseteq \{f \in \mathcal{H} \colon \|f\| \le C\}$, where C is fixed. Note that the range of K restricted to S is also bounded: $\|Kf\| \le \|K\|_{op}\|f\| \le C\|K\|_{op}$. Thus, K maps a bounded set S into a bounded subset of a finite dimensional subspace of $\mathcal{H}$, which is itself precompact. Hence, K is thus compact. □

To describe K explicitly, let $\{\phi_k\}_{k=1}^{n}$ be a basis for $R(K)$. Then, $Kf = \sum_{k=1}^{n} a_k\phi_k$. We want to see how the a_k's depend on f. Consider $\langle Kf, \phi_j\rangle = \langle f, K^*\phi_j\rangle = \sum_{k=1}^{n} a_k\langle\phi_k, \phi_j\rangle$. Next let $\psi_j = K^*\phi_j$, so that $\langle f, K^*\phi_j\rangle = \langle f, \psi_j\rangle$. Because $\{\phi_k\}_{k=1}^{n}$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle\phi_k, \phi_j\rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j\rangle = \sum_{k=1}^{n} G_{j,k}a_k$. Doing so yields $a_k = \sum_{j=1}^{n}(G^{-1})_{k,j}\langle f, \psi_j\rangle$. The a_k's are obviously linear in f. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0, 1]$. A particularly important set of finite rank operators in $\mathcal{C}(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^{n} \phi_k(x)\overline{\psi_k(y)}$, where the functions involved are in L^2. The operator is then $Kf(x) = \int_0^1 k(x, y)f(y)dy$. In the example that we did for resolvents, the kernel was $k(x, y) = xy^2$, and the operator was $Ku(x) = \int_0^1 k(x, y)u(y)dy$. Later, we will show that the Hilbert-Schmidt kernels also yield compact operators. Before, we do so, we will discuss a few more properties of compact operators.

Lemma 2.4. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n. set in $\mathcal{H}$ and let $K \in \mathcal{C}(\mathcal{H})$. Then,* $\lim_{n\to\infty} K\phi_n = 0$.

Proof. Suppose not. Then we may select a subsequence $\{\phi_m\}$ of $\{\phi_n\}_{n=1}^{\infty}$ for which $\|K\phi_m\| \geq \alpha > 0$ for all m. Because K is compact, we can also select a subsequence $\{\phi_k\}$ of $\{\phi_m\}$ for which $\{K\phi_k\}$ is convergent to $\psi \in \mathcal{H}$. Now, $\{\phi_k\}$ being a subsequence of $\{\phi_m\}$ implies that $\|K\phi_k\| \geq \alpha > 0$. Taking the limit in this inequality yields $\|\psi\| \geq \alpha > 0$. Next, note that $\lim_{k\to\infty}\langle K\phi_k, \psi\rangle = \|\psi\|^2$. However, $\lim_{k\to\infty}\langle K\phi_k, \psi\rangle = \lim_{k\to\infty}\langle \phi_k, K^*\psi\rangle = 0$, by Bessel's inequality. Thus, $\|\psi\|^2 = 0$, which is a contradiction. □

This lemma is a special case of a more general result. We say that a sequence $\{f_n\}$ is *weakly convergent* to a $f \in \mathcal{H}$ if and only if for all $g \in \mathcal{H}$ we have $\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle$. For example, the o.n. set in the lemma weakly converges to 0.

There are two important facts concerning weak convergence[1]. The first is that *weakly convergent sequences are bounded* and the second is that *every bounded sequnce has a weakly convergent subsequence.*

Proposition 2.5. *Let $\{f_n\}$ weakly converge to $f \in \mathcal{H}$. If $K \in \mathcal{C}(\mathcal{H})$, then $\lim_{n\to\infty} Kf_n = Kf$. That is, K maps weakly convergent sequences into "strongly" convergent ones.*

Proof. The proof is similar to that of Lemma 2.4. Suppose not. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that $\|Kf_{n_k} - Kf\| \geq \epsilon > 0$. Because $K \in \mathcal{C}(\mathcal{H})$, we may select a subsequence of $\{f_{n_k}\}$, $f_{n_{k_j}} =: \widetilde{f}_j$, such that $K\widetilde{f}_j$ converges to ψ. From the inequality above, we have that $\|\psi - Kf\| \geq \epsilon$. We can use this and the weak convergence of $K\widetilde{f}_j$ to arrive at a contradiction. We leave the details as an exercise. □

We remark that the converse is true, too. This leads to an alternative characterization of compact operators: *K is compact if and only if K maps weakly convergent sequences into strongly convergent ones.* See the book *Functional Analysis*, by F. Riesz and B. Sz.-Nagy.

Our next result is one of the most important theorems in the theory of compact operators.

Theorem 2.6. *$\mathcal{C}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$.*

[1]See Riesz-Nagy, p. 64.

Proof. Suppose that $\{K_n\}_{n=1}^\infty$ is a sequence in $\mathcal{C}(\mathcal{H})$ that converges to $K \in \mathcal{B}(\mathcal{H})$, in the operator norm. We want to show that K is compact. Assume the $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, with $\|v_k\| \le C$ for all k. Compactness will follow if we can prove that $\{Kv_k\}$ has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form $\{K_1 v_k\}$. Since K_1 is compact, we can select a subsequence $\{v_k^{(1)}\}$ such that $\{K_1 v_k^{(1)}\}$ is convergent. We may carry out the same procedure with $\{K_2 v_k^{(1)}\}$, selecting a subsequence of $\{K_2 v_k^{(1)}\}$ that is convergent. Call it $\{v_k^{(2)}\}$. Since this is a subsequence of $\{v_k^{(1)}\}$, $\{K_1 v_k^{(2)}\}$ is convergent. Continuing in this way, we construct subsequences $\{v_k^{(j)}\}$ for which $\{K_m v_k^{(j)}\}$ is convergent for all $1 \le m \le j$. Next, we let $\{u_j := v_j^{(j)}\}$, the "diagonal" sequence. This is a subsequence of all of the $\{v_k^{(j)}\}$'s. Consequently, for n fixed, $\{K_n u_j\}_{j=1}^\infty$ will be convergent. To finish up, we will use an "up, over, and around" argument. Note that for all ℓ, m,

$$\|Ku_\ell - Ku_m\| \le \|Ku_\ell - K_n u_\ell\| + \|K_n u_\ell - K_n u_m\| + \|K_n u_m - Ku_m\|$$

Since $\|Ku_\ell - K_n u_\ell\| \le \|K - K_n\|_{op}\|u_\ell\| \le C\|K - K_n\|_{op}$ and, similarly, $\|Ku_m - K_n u_m\| \le C\|K - K_n\|_{op}$, so $\|Ku_\ell - Ku_m\| \le 2C\|K - K_n\|_{op} + \|K_n u_\ell - K_n u_m\|$. Let $\varepsilon > 0$. First choose N such that for $n \ge N$, $\|K - K_n\|_{op} < \varepsilon/(4C)$. Fix n. Because $\{K_n u_\ell\}$ is convergent, it is Cauchy. Choose N' so large that $\|K_n u_\ell - K_n u_m\| < \varepsilon/2$ for all $\ell, m \ge N'$. Putting these two together yields $\|Ku_\ell - K_n u_\ell\| < \varepsilon$, provided $\ell, m \ge N'$. Thus $\{Ku_\ell\}$ is Cauchy and therefore convergent. □

Corollary 2.7. *Hilbert-Schmidt operators are compact.*

Proof. Let $\mathcal{H} = L^2[0,1]$ and suppose $k(x,y) \in L^2(R)$, $R = [0,1] \times [0,1]$. The associated Hilbert-Schmidt operator is $Ku = \int_0^1 k(x,y)u(y)dy$. Let $\{\phi_n\}_{n=1}^\infty$ be an o.n. basis for $L^2[0,1]$. With a little work, one can show that $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^\infty$ is an o.n. basis[2] for $L^2(R)$. Also, from Proposition 2 in the notes on *Bounded Operators & Closed Subspaces*, we have that $\|K\|_{op} \le \|k\|_{L^2(R)}$. Expand $k(x,y)$ in the o.n. basis $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^\infty$:

$$k(x,y) = \sum_{n,m=1}^\infty \alpha_{m,n}\phi_n(x)\phi_m(y), \ \alpha_{m,n} = \langle k(x,y), \phi_n(x)\phi_m(y)\rangle_{L^2(R)}$$

[2]See Keener, Theorem 3.5

Next, let $k_N(x,y) = \sum_{n,m=1}^{N} \alpha_{m,n}\phi_n(x)\phi_m(y)$ and also K_N be the finite rank operator $K_N u(x) = \int_0^1 k_N(x,y)u(y)dy$. By Parseval's theorem, we have that

$$\begin{aligned}
\|k - k_N\|_{L^2(R)}^2 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_{m,n}|^2 - \sum_{m=1}^{N}\sum_{n=1}^{N} |\alpha_{m,n}|^2 \\
&= \sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty} |\alpha_{m,n}|^2 + \sum_{m=1}^{N}\sum_{n=N+1}^{\infty} |\alpha_{m,n}|^2 \\
&\le \sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty} |\alpha_{m,n}|^2 + \sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty} |\alpha_{m,n}|^2.
\end{aligned}$$

Both terms go to 0 as $N \to \infty$. To make this clear, let $\tilde{a}_m^2 = \sum_{n=1}^{\infty} a_{m,n}^2$. Because $\sum_{m=1}^{\infty} \tilde{a}_m^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_{m,n}|^2$, the series $\sum_{m=1}^{\infty} \tilde{a}_m^2$ is absolutely convergent; consequently, $\lim_{N\to\infty} \sum_{m=N+1}^{\infty} \tilde{a}_m^2 = 0$. Using this for both terms in the inequality implies that $\lim_{N\to\infty} \|k - k_N\|_{L^2(R)}^2 = 0$. As we mentioned above, $\|K - K_N\|_{op} \le \|k - k_N\|_{L^2(R)}$, so

$$\lim_{N\to\infty} \|K - K_N\|_{op} = 0.$$

Thus K is the limit in $\mathcal{B}(L^2[0,1])$ of finite rank operators, which are compact. By Theorem 2.6 above, K is also compact. □

We now turn to some of the algebraic properties of $\mathcal{C}(\mathcal{H})$.

Proposition 2.8. *Let $K \in \mathcal{C}(\mathcal{H})$ and let $L \in \mathcal{B}(\mathcal{H})$. Then both KL and LK are in $\mathcal{C}(\mathcal{H})$.*

Proof. Let $\{v_k\}$ be a bounded sequence in $\mathcal{H}$. Since L is bounded, the sequence $\{Lv_k\}$ is also bounded. Because K is compact, we may find a subsequence of $\{KLv_k\}$ that is convergent, so $KL \in \mathcal{C}(\mathcal{H})$. Next, again assuming $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, we may extract a convergent subsequence from $\{Kv_k\}$, which, with a slight abuse of notation, we will denote by $\{Kv_j\}$. Because L is bounded, it is also continuous. Thus $\{LKv_j\}$ is convergent. It follows that LK is compact. □

Proposition 2.9. *K is compact if and only if K^* is compact.*

Proof. Because K is compact, it is bounded and so is its adjoint K^*, in fact $\|K^*\|_{op} = \|K\|_{op}$. By Proposition 2.8, we thus have that KK^* is compact. It follows that if $\{u_n\}$ is a bounded sequence in $\mathcal{H}$, then we may extract a subsequence of $\{u_n\}$, denoted by $\{v_j\}$, such that $\{KK^*v_j\}$ is convergent. This of course means that this sequence is also Cauchy. Note that

$$\langle KK^*(v_j - v_k), v_j - v_k\rangle = \langle K^*(v_j - v_k), K^*(v_j - v_k)\rangle = \|K^*(v_j - v_k)\|^2.$$

From this and the fact that $\{v_j\}$, being a subsequence of the bounded sequence $\{u_n\}$, is itself bounded, we see that $\langle KK^*(v_j - v_k), v_j - v_k\rangle \le \|v_j - v_k\|\, \|KK^*(v_j - v_k)\| \le C\|KK^*(v_j - v_k)\|$. Thus,

$$\|K^*(v_j - v_k)\|^2 \le C\|KK^*(v_j - v_k)\|$$

Since $\{KK^*v_j\}$ is Cauchy, for every $\varepsilon > 0$, we can find N such that whenever $j, k \ge N$, $\|KK^*(v_j - v_k)\| < \varepsilon^2/C$. It follows that $\|K^*(v_j - v_k)\| < \varepsilon$, if $j, k \ge N$. This implies that $\{K^*v_j\}$ is Cauchy and therefore convergent. □

We want to put this in more algebraic language. Taking L to be compact in Proposition 2.8, we have that the product of two compact operators is compact. Since $\mathcal{C}(\mathcal{H})$ is already a subspace, this implies that it is an algebra. Moreover, by taking L to be just a bounded operator, we have that $\mathcal{C}(\mathcal{H})$ is a two-sided *ideal* in the algebra $\mathcal{B}(\mathcal{H})$. Since K being compact implies K^* is compact, $\mathcal{C}(\mathcal{H})$ is closed under the operation of taking adjoints; thus, $\mathcal{C}(\mathcal{H})$ is a $*$-ideal. Finally, by Theorem 2.6, we have that $\mathcal{C}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$. We summarize these results as follows.

Theorem 2.10. *$\mathcal{C}(\mathcal{H})$ is a closed, two-sided, $*$-ideal in $\mathcal{B}(\mathcal{H})$.*

We remark that a closed *-algebra in $\mathcal{B}(\mathcal{H})$ is called a C*-algebra. So, $\mathcal{C}(\mathcal{H})$ is a C*-algebra that is also a two-sided ideal in $\mathcal{B}(\mathcal{H})$.

Previous: Example of the Fredholm alternative and resolvent

Next: the closed range theorem