Prop. 1 Let $u \in dom(L)$. Then,

$$D(u) = \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j, \quad D(u) = \int_0^1 (p u'^2 + q u^2)\,dx + p(1)\sigma(1)u(1)^2 + p(0)\sigma(0)u(0)^2.$$

Proof: Use remark 3.

2. Courant-Fischer for S-L problems.

Let $\phi_1, \phi_2, \dots, \phi_{n-1}$ be eigenfunct. of $Lu = \lambda u$ corresp. to $\lambda_1, \dots, \lambda_{n-1}$ ($\lambda_k \uparrow$)

Prop. 2
$$\min_{\substack{u \perp \{\phi_1,\dots,\phi_{n-1}\},\\ u \in \text{admissible}}} D(u) = \lambda_n.$$

Proof: If $u \perp \{\phi_1, \dots, \phi_{n-1}\}$, then by our prev. results,

$$D(u) = \sum_{j=n}^{\infty} \lambda_j \alpha_j^2, \quad \alpha_j = \langle u, \phi_j \rangle, \quad \underbrace{\sum \alpha_j^2 = 1}_{H(u)=1}$$

$$\Rightarrow D(u) \geq \lambda_n \Big(\underbrace{\sum_{j=n}^{\infty} \alpha_j^2}_{1} \Big) = \lambda_n.$$

n^{th} eigenvector

But: $D(\phi_n) = \lambda_n$. Thus, the result holds.

Let $V_n = sp\{\psi_1, \dots, \psi_{n-1}\}$, where $\psi_j \in L^2_w[0,1]$. Define

$$d(\psi_1, \dots, \psi_n) = \inf_{\substack{u \in ad.,\ H(u)=1\\ u \perp \{\psi_1, \dots, \psi_{n-1}\}}} D(u).$$

Courant-F Th.
$$\lambda_n = \max_{\{\psi_1, \dots, \psi_{n-1}\}} d(\psi_1, \dots, \psi_n).$$

Proof: By Prop. 2, $\lambda_n = \min_{\substack{u \perp \{\ldots\} \\ u \text{ admissible}}} D(u)$. Thus,

we have $\max_{\{\psi_1, \ldots \psi_{n-1}\}} d(\psi_1, \ldots \}) \geq d(\phi_1, \ldots \phi_{n-1}) = \lambda_n$.

Now, we repeat the proof of C-F for matrices: Expand the ψ_j's in the ϕ_k basis,

$$\psi_j = \sum_{k=1}^{\infty} \underbrace{\langle \psi_j, \phi_k \rangle}_{\beta_{jk}} \phi_k ,$$

also $u = \sum_{k=1}^{\infty} \alpha_k \phi_k$, ~~u is a~~ u is admissible.

Then, $u \perp \{\psi_1 \ldots \psi_{n-1}\} \Rightarrow \langle u, \psi_j \rangle_w = \sum_{k=1}^{\infty} \beta_{jk} \alpha_k = 0, \; j = 1, \ldots, n-1.$

Now, choose $\alpha_{n+1} = \alpha_{n+2} = \ldots = 0$. Then we have

$$\sum_{k=1}^{n} \beta_{jk} \alpha_k = 0 \quad \left\{ \begin{array}{l} j = 1, \ldots, n-1 \Rightarrow \\ n-1 \text{ eqns. in } n \text{ var.} \end{array} \right.$$

$\Rightarrow$ There is a nontriv. soln. for the α's.

Also, $D(u) = \sum_{k=1}^{n} \alpha_k^2 \lambda_k \leq \underbrace{\left(\sum_{k=1}^{n} \alpha_k^2 \right)}_{1} \lambda_n \quad (\lambda_k \leq \lambda_n)$

$\therefore \; D(u) \leq \lambda_n$.

$\Rightarrow \quad \cancel{\min d(\psi_1, \ldots, \psi_{n-1}) \leq} \lambda_n$

$$d(\psi_1, \ldots, \psi_{n-1}) = \min_{u \perp \{\psi_1 \ldots \psi_{n-1}\}} D(u) \leq \lambda_n$$

$\Rightarrow \quad \max d(\ldots) \leq \lambda_n$. By prev. prop, result holds.