HW 7

Problem 7.1.2(a)

$\underbrace{L_1 x = (0, x_1, x_2, \cdots)}_{\text{Right shift}}$, $L_2 x = \underbrace{(x_2, x_3, \cdots, x_k, \cdots)}_{\text{Left shift}}$

In class, we showed that $L_1^* = L_2$, $\|L_1\| = \|L_2\| = 1$, $\sigma(L_2) = \sigma_d + \sigma_c$, where $\sigma_d = \{|\lambda| < \}$ & $\sigma_c = \{|\lambda| = 1\}$. (This implies that $\sigma(L_2) = \{|\lambda| \le 1\}$ & $\rho(L_2) = \{|\lambda| > 1\}$.

1) $L_1 - \lambda I$ is 1:1 for $|\lambda| \le 1$.

$(L_1 - \lambda I) x = (0 - \lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3 \cdots)$

$\Rightarrow -\lambda x_1 = 0,\ x_1 - \lambda x_2 = 0, \cdots, x_{k-1} - \lambda x_k = 0 \cdots$

~~Case 1 $\lambda = 0$. Get $x_{k-1} = 0$,~~

Case 1 $\lambda = 0 \Rightarrow x_1 = 0, x_2 = 0, \cdots, x_k = 0, \cdots$

$\Rightarrow x = 0$, $\therefore L_1$ is 1:1.

Case 2 $\lambda \neq 0$, $|\lambda| \le 1$.

$-\lambda x_1 = 0 \Rightarrow x_1 = 0$

$x_1 - \lambda x_2 = 0 \Rightarrow x_2 = \frac{1}{\lambda} x_1 = 0$

$\cdots\ x_k = \frac{1}{\lambda} x_{k-1} = \frac{1}{\lambda} \cdot 0 = 0, \Rightarrow x = 0$

~~$x = 0$~~

$\therefore L_1 - \lambda I$ is 1:1 for $|\lambda| \le 1$,

2) ~~$\sigma_{res} = \{|\lambda| < 1\}$. If $Rang(L_1 - \lambda I) \neq \ell_2$, then there exists $y \in \ell_2$ s.t. $y \neq 0$, $y \perp Rang(L_1 - \lambda I)$.~~
~~$\Leftrightarrow \forall x \in \ell_2, \langle (L_1 - \lambda I)x, y \rangle = 0 \Rightarrow$~~

2) $\overline{\text{Range}(L_1-\lambda I)} \neq l_2$.

Proof: We want to find $y \in l_2$, $y \neq 0$, s.t. $y \perp \overline{\text{Range}(L_1-\lambda I)}$. To do this, let's solve

$$\langle (L_1-\lambda I)x, y\rangle = 0$$

$$\Rightarrow \langle x, (L_1^*-\bar\lambda)y\rangle = 0$$

$$\Rightarrow L_1^* y = \bar\lambda y \Rightarrow \bar\lambda \in \sigma_d(L_1^*) = \sigma_d(L_2).$$

Since $|\lambda|<1$ & $\sigma_d(L_2) = \{|\lambda|<1\}$, $\bar\lambda \in \sigma_d(L_2)$ & y is an eigenvector corresp. to $\bar\lambda$ for L_2. Thus, $\overline{\text{Range}(L_1-\lambda I)} \neq l_2$.

3) $\overline{\text{Range}(L_1-\lambda I)} = l_2$ if $|\lambda|=1$. By the same argument we used in 2), we have $\overline{\text{Range}(L_1-\lambda I)} \neq l_2$ ~~iff~~ if & only if $\bar\lambda \in \sigma_d(L_2)$. But, $|\bar\lambda|=1 \Rightarrow \bar\lambda \notin \sigma_2 \Rightarrow \overline{\text{Range}(L_1-\lambda I)} = l_2$.

4) $\sigma_{resid}(L_1) = \{|\lambda|<1\}$, $\sigma_c(L_1) = \{|\lambda|=1\}$, $\sigma_d = \emptyset$.

The first follows ~~fr~~ from $L_1-\lambda I$ being 1:1 and $\overline{\text{Range}(L_1-\lambda I)} \neq l_2$ iff $|\lambda|<1$. The second follows from three things:

(i) $L_1-\lambda I$ is 1:1.

(ii) $\sigma(L_1) = \{|\lambda|\le 1\}$. ($\rho(L_1) \supset \{|\lambda|>1\}$ but $\sigma(L_1) \subset \{|\lambda|<1\}$. $\sigma(L_1)$ being closed $\Rightarrow \sigma(L_1) = \{|\lambda|\le 1\}$.)

(iii) $\text{Range}(L_1-\lambda I)$ is dense in l_2.

$$\Rightarrow \sigma_c(L_1) = \{|\lambda|=1\}.$$