Problem 4, §8.2

Recall that we derived the sine transform $\widehat{G}(\mu,t)$ of the Green's function $G(x,t)$ for the wave equation with boundary conditions $G(0,t)=0$ and initial conditions $G(x,0)=0$, $G_t(x,0)=0$ for $x\ge 0$. (See problem 4, §8.2). What we found was that

$$\widehat{G}(\mu,t)=H(t-\tau)\frac{\sin(\mu\xi)\sin(\mu(t-\tau))}{\mu}.$$

The function $\widehat{G}(\mu,t)=0$ when $t-\tau\le 0.$, and we multiply by the Heaviside step function $H(t-\tau)$ to take care of this. Note that although the solution is a tempered distribution, for t fixed it is in $L^2([0,\infty))$ as a function of μ. (The integral is improper, but convergent.) Thus the formula for the inverse sine transform,

$$G(x,t)=\frac{2H(t-\tau)}{\pi}\int_0^\infty\frac{\sin(\mu\xi)\sin(\mu(t-\tau))}{\mu}\sin(\mu x)d\mu,$$

applies in the usual L^2 sense. Using the product-to-sum formulas from trigonometry, we obtain

$$\begin{aligned}G(x,t)=&\frac{H(T)}{2\pi}\int_0^\infty\frac{\sin(\mu(T+x-\xi))}{\mu}d\mu+\frac{H(T)}{2\pi}\int_0^\infty\frac{\sin(\mu(T-x+\xi))}{\mu}d\mu\\&-\frac{H(T)}{2\pi}\int_0^\infty\frac{\sin(\mu(T-x-\xi))}{\mu}d\mu-\frac{H(T)}{2\pi}\int_0^\infty\frac{\sin(\mu(T+x+\xi))}{\mu}d\mu,\end{aligned}$$

where $T:=t-\tau$. Each integral has the form $\int_0^\infty\frac{\sin(a\mu)}{\mu}d\mu$, where a is real. We amy assume $a\ne 0$. Changing variables from μ to $\nu=|a|\mu$ then yields

$$\int_0^\infty\frac{\sin(a\mu)}{\mu}d\mu=\text{sign}(a)\underbrace{\int_0^\infty\frac{\sin(\nu)}{\nu}d\nu}_{\pi/2}=\frac{\pi}{2}\text{sign}(a).$$

It follows that $G(x,t)$ has the form

$$\begin{aligned}G(x,t)=&H(T)\big(\text{sign}(T+x-\xi)+\text{sign}(T-x+\xi)\big)/4\\&-H(T)\big(\text{sign}(T-x-\xi)-\text{sign}(T+x+\xi)\big)/4.\end{aligned}$$

Checking various sign combinations, one can show that this expression has the equivalent form

$$G(x,t)=\frac{1}{2}H(t-\tau-|x-\xi|)-\frac{1}{2}H(t-\tau-|x+\xi|).$$

This agrees with the solution in §8.2 that was found via the "method of images."