Test 1

Part 1. . This take-home part of Test 1 is due Friday, 23 March. You may consult any written or online source. You may *not* consult any person, either a fellow student or faculty member, except your instructor

1. **(10 pts.)** Using cubic polynomials, approximate the second eigenvalue of $u'' + \lambda u = 0$, $u(0) = 0$, $u(1) + u'(1) = 0$.

2. **(10 pts.)** Let $w = f(z)$ be analytic in a region containing the disk $|z| \le 1$, and suppose that $f(0) = 0$, $f'(0) \ne 0$. For z small enough, $f(z)$ maps this disk one-to-one and onto a region in the w plane containing a disk $|w| \le a$. Show that the function inverse to f, $g(w)$, is given by the contour integral

$$g(w) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z f'(z)}{f(z) - w} dz.$$

   For $f(z) = e^z - 1$ and $|w|$ small, expand the integrand in a power series in w. Calculate the coefficients in this series and verify that the result is $g(w) = \log(1 + w)$, where the log uses the principal branch.

3. **(5 pts.)** Problem 12, pg. 280. In addition to the hint given, look at pg. 265 for the behavior of $H_0^{(1)}(z)$ for large z.

4. **(5 pts.)** Problem 23, pg. 281. (See §5.2.1.)

5. **(10 pts.)** The Hermite polynomials $H_n(x)$ satisfy the recurrence relation,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \;\; H_0(x) = 1 \text{ and } H_1(x) = 2x.$$

   Use this to show that the generating function for the Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2tx - t^2}.$$

6. **(10 pts.)** Use your favorite software to plot $\sqrt{\frac{\pi x}{2}} J_1(x)$ and $\sqrt{\frac{\pi x}{2}} Y_1(x)$. Determine approximately when the asymptotic formulas for these quantities hold. (Plot these for various ranges of x, starting at $x = 0.5$.)