Midterm Test

Instructions: *This test is due Thursday, 20 March. You may consult any written or online source. You may* not *consult any person, either a fellow student or faculty member, except me.*

1. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}.$$

   (a) **(5 pts.)** Show that L is self-adjoint.

   (b) **(10 pts.)** Let z be in $\mathbb{C} \setminus [0,\infty)$. Find the Green's function $G(x,\xi;z)$ for $-G'' - zG = \delta(x-\xi)$, with $G'(0,\xi;z) = 0$.

2. A mass m is attached to a pendulum of length ℓ and negligible weight. The pendulum itself is attached to a fixed pivot and allowed to swing freely, with the mass subject only to gravity.

   (a) **(5 pts.)** Take the pivot to be the origin. Use spherical coordinates to write the Lagrangian for the system. (The angle θ is the colattitude and the angle ϕ is the longitude.)

   (b) **(10 pts.)** Find the Hamiltonian for the system along with two constants of the motion. Use these to find a first order nonlinear differential equation for θ.

3. **(10 pts.)** Using cubic polynomials, approximate the second eigenvalue of $u'' + \lambda u = 0$, $u(0) = 0$, $u(1) + u'(1) = 0$.

4. Let $w = f(z)$ be analytic in a region containing the disk $|z| \le 1$, and suppose that $f(0) = 0$, $f'(0) \ne 0$. For z small enough, $f(z)$ maps this disk one-to-one and onto a region in the w plane containing a disk $|w| \le a$.

   (a) **(10 pts.)** Show that the function inverse to f, $g(w)$, is given by the contour integral

$$g(w) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z f'(z)}{f(z) - w} dz. \tag{1}$$

(b) **(5 pts.)** For $f(z) = (z-2)^2 - 4$ and $|w|$ small, expand the integrand in (1) in a power series in w. Calculate the coefficients in this series and verify that the result is $z = 2 + \sqrt{w+4}$, where the square root uses principal branch in which $\arg(z) \in (-\pi, \pi]$.

5. **(15 pts.)** Recall that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, which is valid when z is in the right half plane, $\Re(z) > 0$. Apply the dominated convergence theorem to the difference quotient $(\Gamma(z+h) - \Gamma(z))/h$ to show that $\Gamma'(z) = \int_0^\infty t^{z-1} \ln(t) e^{-t} dt$.

6. **(15 pts.)** Show that for any real u this holds:

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x-u)dx = \operatorname{sinc}(u).$$

7. **(15 pts.)** Let $\alpha > 0$, $0 < \beta < 1$, and $\mu > 0$. Show that

$$\int_{-\infty}^{\infty} \frac{e^{-i\mu x}}{(x+i\alpha)^\beta} dx = 2e^{-\alpha\mu - \pi i\beta/2} \sin(\pi\beta/2)\Gamma(1-\beta),$$

where z^β has $-\pi/2 < \arg(z) \le 3\pi/2$. (Hint: there is a branch cut for $(z+i\alpha)^\beta$ along the imaginary axis $\Im(z) = y$ starting at $y = -\alpha$ and running down to $y = -\infty$. Deform the contour to make use of the cut.)