Midterm

This test due Wednesday, 3/19/2014. You may consult any written or online source. You may *not* consult anyone, except your instructor

1. **(10 pts.)** A mass m is attached to a pendulum of length ℓ and negligible weight. The pendulum itself is attached to a fixed pivot and allowed to swing freely, with the mass subject only to gravity. (Gravity points in the $-\mathbf{k}$ direction). Take the pivot to be the origin. Using spherical coordinates, where θ is the colatitude (off the direction $\mathbf{k}$) and ϕ is the longitude, find the Hamiltonian for the system, along with two constants of the motion. Use these to find a first order nonlinear differential equation for θ.

2. **(15 pts.)** Let $f(x)$ be continuous on $[a, b]$. Suppose that, for all $\eta \in C^k[a, b]$ satisfying $\eta^{(j)}(a) = \eta^{(j)}(b) = 0$, $j = 0, \ldots, k-1$, we have $\int_a^b f(x)\eta^{(k)}(x)dx = 0$. Show that $f(x)$ is a polynomial of degree $k-1$.

3. **(15 pts.)** Let $J[y] = \int_0^1 y^{(k)}(x)^2 dx$. The admissible set for J consists of all piecewise C^k curves for which $y(j/n) = y_j$, $j = 0, \ldots n$, with the discontinuities in $y^{(k)}$ occurring only at the points $x_j = j/n$. Use the previous problem to show that the minimizer $y(x)$ for J is in the finite element space $S^{1/n}(2k-1, 2k-2)$.

4. Let f and g be analytic functions in a neighborhood of $z = 0$. Suppose that the Taylor series expansions for f and g are $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, respectively.

   (a) **(5 pts.)** Show that the coefficients in the power series for $f(z)g(z)$ are $c_n = \sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} a_k b_{n-k}$.

   (b) **(5 pts.)** If $a_0 \neq 0$, then $\frac{1}{f(z)}$ is analytic at $z = 0$. Show that if $\frac{1}{f(z)} = \sum_{k=0}^{\infty} b_k z^k$, then $b_0 = \frac{1}{a_0}$ and the remaining b's satisfy the recurrence relation

   $$b_n = -\sum_{k=0}^{n-1} \frac{a_{n-k}}{a_0} b_k, \ n \geq 1.$$

(c) **(5 pts.)** Given that $\frac{e^z-1}{z} = \sum_{k=0}^{\infty} \frac{z^n}{(n+1)!}$, find a recurrence relation for the B_k's in the series

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k z^k.$$

In addition, find B_k for $0 \le k \le 5$. (The B_k's are called the Bernoulli numbers.)

(d) **(10 pts.)** Let $S_m(n) = \sum_{k=1}^{n} k^m$. Show that $1+\sum_{m=0}^{\infty} S_m(n) \frac{z^m}{m!} = \sum_{k=0}^{n} e^{kz} = \frac{e^{(n+1)z}-1}{z} \frac{z}{e^z-1}$. Find a formula for $S_m(n)$ in terms of B_k's, $k \le m$. Use it to find $S_5(n)$.

5. **(10 pts.)** Let $I((\lambda) = \int_{-\infty}^{\infty} e^{-x^2+i\lambda x} dx$, where $\lambda > 0$. You are given $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. In addition, let C_R be the counter clockwise oriented boundary of the rectangle with vertices $-R, R, R+\frac{1}{2}i\lambda, -R+\frac{1}{2}i\lambda$. Integrate e^{-z^2} around C_R and let $R \to \infty$ to show that $I(\lambda) = \sqrt{\pi} e^{-\lambda^2/4}$.

6. **(10 pts.)** For $-1 < \beta < 1$, let $I(\beta) = \int_0^{\infty} \frac{x^\beta}{(1+x)^2} dx$. Use a "keyhole" contour to find $I(\beta)$.

7. **(15 pts.)** The following is a special case of the Paley-Wiener Theorem. Let $f(z)$ be an entire function that satisfies these conditions: (1) for $x \in \mathbb{R}$, $f(x) \in L^1(\mathbb{R})$; (2) there exist constants $A > 0$, $\rho > 0$, and $\delta > 0$ such that $|f(z)| \le A(|z|+1)^{-\delta} e^{\rho|\mathrm{Im}(z)|}$ for all $z \in \mathbb{C}$. Show that for all $\xi \in \mathbb{R}$ such that $|\xi| > \rho$ one has that

$$\int_{-\infty}^{\infty} f(x) e^{i\xi x} dx = 0.$$