Test 1

Take-home part. This take-home part of the Midterm is due Tuesday, 3/24/2015. You may consult any written or online source. You may *not* consult anyone, except your instructor

1. **(10 pts.)** Let $f(x)$ be continuous on $[a,b]$ and suppose that, for all $\eta \in C^{k+1}[a,b]$ satisfying $\eta^{(j)}(a) = \eta^{(j)}(b) = 0$, $j = 0,\ldots,k$, we have $\int_a^b f(x)\eta^{k+1}(x)dx = 0$. Show that $f(x)$ is a polynomial of degree k.

2. **(15 pts.)** Let $\alpha > 0$, $0 < \beta < 1$, and $\mu > 0$. Show that

$$\int_{-\infty}^{\infty} \frac{e^{-i\mu x}}{(x+i\alpha)^\beta} dx = 2e^{-\alpha\mu - \pi i\beta/2} \sin(\pi\beta)\mu^{\beta-1}\Gamma(1-\beta),$$

where z^β has $-\pi/2 < \arg(z) \le 3\pi/2$. (Hint: there is a branch cut for $(z+i\alpha)^\beta$ along the imaginary axis, starting at $y = -\alpha$ and running down to $y = -\infty$. Deform the contour to make use of the cut.)

3. A planet moving around the Sun in an elliptical orbit, with eccentricity $0 \le \varepsilon < 1$ and period P, has time and angle related in the following way. Let $\tau = (2\pi/P)(t - t_p)$, where t_p is the time when the planet is at perihelion – i.e., it is nearest the Sun. Let θ be the usual polar angle and let u be an angle related to θ via

$$(1-\varepsilon)^{1/2}\tan(u/2) = (1+\varepsilon)^{1/2}\tan(\theta/2).$$

It turns out that $\tau = u - \varepsilon\sin(u)$. All three variables θ, u, and τ are measured in radians. They are called the true, eccentric, and mean anomalies, respectively. (*Anomaly* is another word for angle.)

   (a) **(5 pts.)** For $0 \le \varepsilon < 1$, show that the equation $\tau = u - \varepsilon\sin(u)$ may be solved, at least in principle, for $u = u(\tau)$, for all τ Also, show $u(\tau)$ is odd, and that $g(\tau) = u(\tau) - \tau$ is a 2π periodic function of τ. Show that the Fourier series of $g(\tau)$ is a sine series. That is,

$$g(\tau) = \sum_{n=1}^{\infty} b_n \sin(n\tau).$$

(b) **(10 pts.)** Show that $b_n = (2/n)J_n(n\varepsilon)$, $n = 1, 2, \ldots$,where J_n is the n^{th} order Bessel function of the first kind. Thus, we have that

$$u = \tau + \sum_{n=1}^{\infty}(2/n)J_n(n\varepsilon)\sin(n\tau).$$

4. Consider the set orthogonal polynomials $h_n(x)$ generated via the Gram-Schmidt process with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx.$$

(a) **(5 pts.)** Show that the polynomial $H_n(x) := (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$ satisfies $\langle p, H_n\rangle = 0$ for all polynomials of degree $n-1$ or less. Explain why this implies that H_n is, up to a constant factor, h_n.

(b) **(5 pts.)** By the Cauchy integral formula for derivatives, we have that

$$\frac{d^n}{dz^n}(e^{-z^2}) = \frac{n!}{2\pi i}\oint_C \frac{e^{-\zeta^2}}{(\zeta - z)^{n+1}}d\zeta,$$

where C is any simple closed contour containing z in its interior. Use this show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\frac{d^n}{dz^n}(e^{-z^2}) = e^{-(z-t)^2},$$

and, from the definition of the H_n's, that

$$\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = e^{2tz-t^2},$$

which is the generating function for the Hermite polynomials. (The Hermite polynomials here are the ones that are used in physics.)